

# Analytical Expressions for the Electromagnetic Dyadic Green's Function in Graphene and Thin Layers

Alexey Yu Nikitin, Francisco J. Garcia-Vidal, and Luis Martin-Moreno

(Invited Paper)

**Abstract**—An analytical general analysis of the electromagnetic dyadic Green's function for 2-D sheet (or a very thin film) is presented, with an emphasis on the case of graphene. A modified steepest descent treatment of the fields from a point dipole given in the form of Sommerfeld integrals is performed. We sequentially derive the expressions for both out-of-plane and in-plane fields of both polarizations. It is shown that the analytical approximation provided is very precise in a wide range of distances from a point source down to a deep subwavelength region (1/100 of wavelength). We separate the contribution from the pole, the branch point, and discuss their interference. The asymptotic expressions for the fields are composed of the plasmon, Norton wave, and the components corresponding to free space.

**Index Terms**—Dyadic Green's function (DGF), graphene, plasmon, thin films.

## I. INTRODUCTION

ELECTROMAGNETIC properties of graphene have recently received a lot of attention due to a variety of application in photonics [1]. One of the attractive properties of graphene is its capacity to support highly localized (nanometric) surface modes, i.e., graphene surface plasmons (GSPs) in terahertz (THz) and microwave frequency ranges [2]–[12]. GSPs and their gate tunability open interesting possibilities for construction of tunable metamaterials [4], and merging photonics and electronics. In particular, for applications related to strong light–matter interactions and biosensing, the interaction of a graphene sheet with a point emitter presents a special interest [5]–[12]. Localized excitation has allowed experimental demonstration of GSPs [11], [12]. In comparison with other experimental techniques, local excitation of GSPs is more favorable due to very high values of GSP momentums.

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A. Y. Nikitin and L. Martin-Moreno are with the Instituto de Ciencia de Materiales de Aragón and Departamento de Física de la Materia Condensada, Consejo Superior de Investigaciones Científicas, Universidad de Zaragoza, E-50009 Zaragoza, Spain (e-mail: alexeynik@rambler.ru; lmm@unizar.es).

F. J. Garcia-Vidal is with the Departamento de Física Teórica de la Materia Condensada, Universidad Autónoma de Madrid, E-28049 Madrid, Spain (e-mail: fj.garcia@uam.es).

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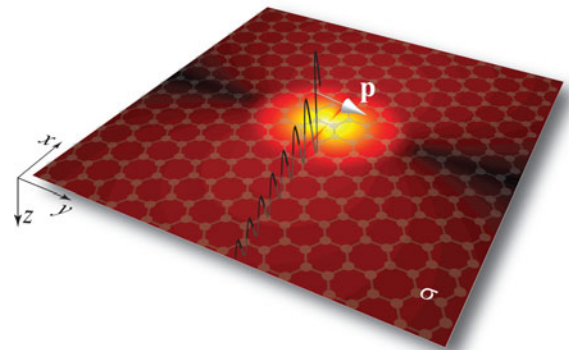


Fig. 1. Schematic of the studied system. A point source (dipole) is placed on a graphene sheet, characterized by a 2-D conductivity  $\sigma$ . The color plot represents an example of the spatial distribution of the electric field modulus for shown orientation of the dipole moment. Representative parameters for the graphene sheet corresponding to the formation of GSP are (see Appendix B for the definitions):  $\Omega = 0.4$ ,  $T = 300$  K,  $\tau = 1$  ps, and  $\mu = 0.2$  eV.

The computation of patterns of the electromagnetic fields in graphene created by point source (as well as spontaneous emission rates) requires the knowledge of the dyadic Green's function (DGF) [2], [3], [8]–[10]. Its calculation involves notoriously difficult Sommerfeld-type integrals, with the integrands containing quickly oscillating functions, poles, and branch cuts [13]–[15].

In this paper, we will perform an analytical analysis of DGF, providing an asymptotic series expansion with the modified steepest descent method [13], [15]. We will show that this expansion, while being exact for long distances, in practice is very precise outside of its formal validity range. We will explicitly provide contributions for GSP, out-of-plane propagating waves, and field components decaying algebraically along the graphene sheet. Notice that while all the examples correspond to the conductivity of graphene, the analytical expressions are applicable to any 2-D sheet. The analytical expression can be useful for treatment of more complicated problems, related, for instance, to Lippmann–Schwinger integral equation [14].

## II. FORMULATION OF THE PROBLEM

Let us consider a dipole with an arbitrary oriented dipole moment  $\mathbf{p}$  placed at the point  $(0, 0, Z')$ , with  $Z'$  being the distance from a free-standing graphene sheet which covers the plane  $Z = 0$  (see Fig. 1). Without loss of generality, we will suppose that  $Z' < 0$ . The time dependence is supposed to be  $e^{-i\omega t}$ ,

where  $\omega$  is the angular frequency. Throughout this paper, we express coordinates in the in-plane ( $\mathbf{R}$ ) and normal ( $Z$ ) directions to the graphene sheet in dimensionless units as  $\mathbf{r} = k_\omega \mathbf{R}$  and  $z = k_\omega Z$ , with  $k_\omega = 2\pi/\lambda$  being the free-space (FS) wave vector.

Graphene is represented by its in-plane complex conductivity  $\sigma$ . Throughout this paper, all examples are performed for  $\sigma$  based on the random-phase-approximation [16]–[18] (see Appendix B).

The electric field  $\mathbf{E}(\mathbf{r}, z)$  emitted by our electric dipole is given through the DGF  $\hat{G}(\mathbf{r}, z, z') = \hat{G}(\mathbf{r}, z; \mathbf{r}' = 0, z')$  by the following relation:

$$\mathbf{E}(\mathbf{r}, z) = \hat{G}(\mathbf{r}, z; z') \mathbf{p}(z'). \quad (1)$$

In the next section, we will provide the exact general expressions for  $\hat{G}(\mathbf{r}, z; z')$ .

### III. GENERAL FORM OF GREEN'S DYADIC

We have performed the analysis for a graphene placed on the boundary of two different dielectrics. However, we have found that there is no much qualitative difference between this general case and free-standing (suspended) graphene. Since the analytical formula in the general case is quite lengthy, in this paper we will consider the DGF for a suspended graphene (taking also into account that free-standing samples are widely used in the experiments).

DGF satisfy the following differential equation:

$$\nabla \times \nabla \times \hat{G}(\mathbf{r}, z; z') - k_\omega^2 \hat{G}(\mathbf{r}, z; z') = \hat{1} \delta(\mathbf{r}) \delta(z - z') \quad (2)$$

where  $\hat{1}$  is a diagonal unit matrix and  $\delta$  is the Dirac delta function. The DGF must be complemented by the boundary conditions at the graphene sheet that we present in the following.

#### A. Angular Representation of DGF in Cartesian Coordinates

The solution for  $\hat{G}$  can be expressed (see, e.g., [13] and [14]) in terms of plane waves in FS  $\mathbf{u}_{\mathbf{q}\tau} e^{i\mathbf{q}\mathbf{r} + iq_z z}$ , characterized by their in-plane momentum  $\mathbf{q} = \mathbf{k}/k_\omega$  and polarization  $\tau = \text{TE}, \text{TM}$ ). The unitary vectors characterizing the polarization of each mode are

$$\mathbf{u}_{\mathbf{q}\text{TE}}^\pm = \frac{1}{q} \begin{pmatrix} -q_y \\ q_x \\ 0 \end{pmatrix} \quad \mathbf{u}_{\mathbf{q}\text{TM}}^\pm = \frac{q_z}{q} \begin{pmatrix} q_x \\ q_y \\ \mp \frac{q^2}{q_z} \end{pmatrix} \quad (3)$$

where “+” (“−”) upward (downward) propagation along  $z$ , respectively.  $q_z = \sqrt{1 - q^2}$  is the normalized  $z$ -component of the wave vectors.

The DGF reads

$$\begin{aligned} \hat{G}(\mathbf{r}, z; z') &= \hat{G}_0(\mathbf{r}, z; z') + \hat{G}_R(\mathbf{r}, z; z'), \quad z' < 0, \quad z < 0 \\ \hat{G}(\mathbf{r}, z; z') &= \hat{G}_T(\mathbf{r}, z; z'), \quad z' < 0, \quad z > 0 \end{aligned} \quad (4)$$

with  $\hat{G}_0$  being the DGF in FS

$$\hat{G}_0(\mathbf{r}, z; z') = \sum_{\tau} \int \frac{d\mathbf{q}}{2q_z} \mathbf{u}_{\mathbf{q}\tau}^+ \mathbf{u}_{\mathbf{q}\tau}^{+T} e^{i\mathbf{q}\mathbf{r} + iq_z |z - z'|} \quad (5)$$

and  $\hat{G}_R$  and  $\hat{G}_T$ , respectively, being the contributions due to the reflection and transmission in our 2-D system

$$\begin{aligned} \hat{G}_R(\mathbf{r}, z; z') &= \sum_{\tau} \int \frac{d\mathbf{q}}{2q_z} R_q^{\tau} \mathbf{u}_{\mathbf{q}\tau}^- \mathbf{u}_{\mathbf{q}\tau}^{+T} e^{i\mathbf{q}\mathbf{r} - iq_z (z + z')} \\ \hat{G}_T(\mathbf{r}, z; z') &= \sum_{\tau} \int \frac{d\mathbf{q}}{2q_z} T_q^{\tau} \mathbf{u}_{\mathbf{q}\tau}^+ \mathbf{u}_{\mathbf{q}\tau}^{+T} e^{i\mathbf{q}\mathbf{r} + iq_z (z - z')}. \end{aligned} \quad (6)$$

In these expressions, the superscript “ $T$ ” stands for transposition.

In the aforementioned expressions,  $R_q^{\tau}$  and  $T_q^{\tau}$  are the reflection and transmission coefficients for graphene, respectively. These coefficients can be found by matching the magnetic  $\mathbf{H}$  and electric  $\mathbf{E}$  fields through the boundary conditions<sup>1</sup>

$$\begin{aligned} \mathbf{e}_z \times (\mathbf{E}_- - \mathbf{E}_+) &= 0 \\ \mathbf{e}_z \times (\mathbf{H}_- - \mathbf{H}_+) &= \frac{4\pi}{c} \mathbf{j} = -\frac{4\pi}{c} \sigma \mathbf{e}_z \times [\mathbf{e}_z \times \mathbf{E}_+] \end{aligned} \quad (7)$$

where  $\mathbf{E}_-$  ( $\mathbf{H}_-$ ) and  $\mathbf{E}_+$  ( $\mathbf{H}_+$ ) stay for the electric (magnetic) fields in the regions of negative and positive  $z$ , respectively, and  $\mathbf{e}_z$  is the unitary vector along the  $+z$  direction. As a result of the matching, we have

$$\begin{aligned} R_q^{\text{TE}} &= \frac{-\alpha}{\alpha + q_z} & R_q^{\text{TM}} &= \frac{-\alpha q_z}{\alpha q_z + 1} \\ T_q^{\text{TE}} &= \frac{q_z}{\alpha + q_z} & T_q^{\text{TM}} &= \frac{1}{\alpha q_z + 1} \end{aligned} \quad (8)$$

with  $\alpha = 2\pi\sigma/c$  being the dimensionless 2-D conductivity.

Explicitly, we have for  $\hat{G}_0 = \hat{G}_0^{\text{TM}} + \hat{G}_0^{\text{TE}}$

$$\begin{aligned} \hat{G}_0^{\text{TE}}(\mathbf{r}) &= \frac{ik_\omega}{8\pi^2} \int \frac{d\mathbf{q}}{q_z q^2} e^{i\mathbf{q}\mathbf{r} + iq_z |z - z'|} \\ &\quad \times \begin{pmatrix} q_y^2 & -q_x q_y & 0 \\ -q_x q_y & q_x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \hat{G}_0^{\text{TM}}(\mathbf{r}) &= \frac{ik_\omega}{8\pi^2} \int \frac{d\mathbf{q}}{q^2} e^{i\mathbf{q}\mathbf{r} + iq_z |z - z'|} \\ &\quad \times \begin{pmatrix} q_x^2 q_z & q_x q_y q_z & \mp q_x q^2 \\ q_x q_y q_z & q_y^2 q_z & \mp q_y q^2 \\ \mp q_x q^2 & \mp q_y q^2 & q^4 / q_z \end{pmatrix}. \end{aligned} \quad (9)$$

Analogously,  $\hat{G}_R = \hat{G}_R^{\text{TM}} + \hat{G}_R^{\text{TE}}$  reads

$$\begin{aligned} \hat{G}_R^{\text{TE}}(\mathbf{r}) &= \frac{ik_\omega}{8\pi^2} \int \frac{d\mathbf{q}}{q_z q^2} R_q^{\text{TE}} e^{i\mathbf{q}\mathbf{r} - iq_z (z' + z)} \\ &\quad \times \begin{pmatrix} q_y^2 & -q_x q_y & 0 \\ -q_x q_y & q_x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \hat{G}_R^{\text{TM}}(\mathbf{r}) &= \frac{ik_\omega}{8\pi^2} \int \frac{d\mathbf{q}}{q^2} R_q^{\text{TM}} e^{i\mathbf{q}\mathbf{r} - iq_z (z' + z)} \end{aligned}$$

<sup>1</sup>These boundary conditions connect the magnetic field at different sides of the conducting sheet with the surface electrical current which, in turn, is proportional to the electric field at the sheet. This is different from the approximate impedance boundary conditions in metal–dielectric interfaces, which provide the relation between the magnetic and electric fields only for the dielectric half-space, assuming that the fields decay very rapidly inside the metal (see, e.g., [19]).

$$\times \begin{pmatrix} q_x^2 q_z & q_x q_y q_z & -q_x q^2 \\ q_x q_y q_z & q_y^2 q_z & -q_y q^2 \\ q_x q^2 & q_y q^2 & -q^4/q_z \end{pmatrix} \quad (10)$$

and finally, for the transmission part  $\hat{G}_T = \hat{G}_T^{\text{TM}} + \hat{G}_T^{\text{TE}}$ , we have

$$\begin{aligned} \hat{G}_T^{\text{TE}}(\mathbf{r}) &= \frac{ik_\omega}{8\pi^2} \int \frac{d\mathbf{q}}{q_z q^2} T_q^{\text{TE}} e^{i\mathbf{q}\mathbf{r} + iq_z(z'+z)} \\ &\times \begin{pmatrix} q_y^2 & -q_x q_y & 0 \\ -q_x q_y & q_x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \hat{G}_T^{\text{TM}}(\mathbf{r}) &= \frac{ik_\omega}{8\pi^2} \int \frac{d\mathbf{q}}{q^2} T_q^{\text{TM}} e^{i\mathbf{q}\mathbf{r} + iq_z(z'+z)} \\ &\times \begin{pmatrix} q_x^2 q_z & q_x q_y q_z & -q_x q^2 \\ q_x q_y q_z & q_y^2 q_z & -q_y q^2 \\ -q_x q^2 & -q_y q^2 & q^4/q_z \end{pmatrix}. \quad (11) \end{aligned}$$

Equations (9)–(11) present the angular representation of DGF in Cartesian coordinates. These expressions can be greatly simplified in cylindrical coordinate system.

### B. DGF in Cylindrical Coordinates

As was previously shown, the Purcell factor (the total decay rate normalized to the FS decay rate) of the point emitter placed directly over the graphene monolayer diverges due to losses through the evanescent waves with large  $q$ -components [6], [7], [10]. This means that the dipole placed directly on the graphene surface is quenched. However, the parameter that accounts for the efficiency of the coupling to GSP ( $\beta$ -factor, defined as the ratio of the emitter's decay rate through GSP to its total decay rate) has an optimum value in the region of very small distances from the dipole to graphene:  $|z'_{\text{opt}}|/\lambda \sim 10^{-2}$  (see [10]). Then, taking into account that the problems related to the high values of  $\beta$ -factor are relevant, the range of small distances presents a special interest. Another important point is that the field patterns at the distances  $r > |z'|$  do not differ essentially upon the field patterns created by a dipole lying directly on the monolayer. Therefore, for the aforementioned two reasons, in this paper we will consider the dipole placed directly on the graphene sheet.

We would like to notice that some physical systems can be reduced to a problem of a dipole lying directly on the graphene sheet. For instance, a subwavelength aperture in graphene sheet can be represented by an effective dipole placed directly on the sheet (for comparison with the case of metal films, see [20] and [21]).

Additionally, since the analytical treatment of both reflection ( $z < 0$ ) and transmission ( $z > 0$ ) parts of the DGF is similar, we will derive the expressions for the transmission part  $\hat{G}_T$ .

So, supposing that  $z' = 0^-$  in the previous expressions, Green's dyadic  $\hat{G}(\mathbf{r}, z) \equiv \hat{G}_T(\mathbf{r}, z, z' = 0^-)$  can be simplified. The symmetry of the problem makes it convenient to work in cylindrical coordinates  $(r, \phi, z)$  [see Fig. 2(a)]

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = z. \quad (12)$$

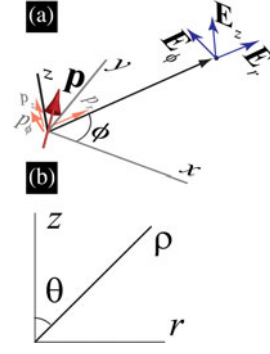


Fig. 2. (a) and (b) Transformation of the coordinates.

In this system of coordinates, Green's dyadic can be obtained from the one in Cartesian coordinates through

$$\hat{G}^{\text{cyl}} = \hat{T}^{-1} \hat{G}^{\text{cart}} \hat{T} \quad (13)$$

where

$$\hat{T} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (14)$$

Having performed this transformation, the DGF in cylindrical coordinates is expressed in the form of Sommerfeld integrals as

$$\hat{G} = \hat{G}_p + \hat{G}_s$$

$$\begin{aligned} \hat{G}_s(r, z) &= \frac{ik_\omega}{8\pi} \int_0^\infty \frac{dq q}{\alpha + q_z} e^{iq_z z} \\ &\times \begin{pmatrix} J_+(qr) & 0 & 0 \\ 0 & J_-(qr) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \hat{G}_p(r, z) &= \frac{ik_\omega}{8\pi} \int_0^\infty \frac{dq}{\alpha q_z + 1} e^{iq_z z} \\ &\times \begin{pmatrix} qq_z J_-(qr) & 0 & -2iq^2 J_1(qr) \\ 0 & qq_z J_+(qr) & 0 \\ -2iq^2 J_1(qr) & 0 & \frac{2q^3}{q_z} J_0(qr) \end{pmatrix} \quad (15) \end{aligned}$$

where the subscripts “s” and “p” correspond to TE and TM polarizations, respectively. In these expressions,  $J_\pm(qr) = J_0(qr) \pm J_2(qr)$  and  $J_n(qr)$  are Bessel functions of  $n$ th order. Equation (15) can be treated numerically. We show some recipes for the integration in the complex plane  $q$  in Appendix A. Let us now proceed with the asymptotic expansion of the DGF.

### IV. ASYMPTOTIC EXPANSION OF THE DGF

In this section, we will derive explicit asymptotic expressions for the elements of DGF following the steepest descent method, modified in order to take into account the presence of both poles and branch points close to the integration path [13], [20].

First, using the identity

$$2J_n = H_n^{(1)} + H_n^{(2)} \quad (16)$$

where  $H_n$  are Hankel functions, we can extend the limit of integration to the whole real  $q$ -axis in (15)

$$\int_0^\infty dq J_n(qr)F(q) = \frac{1}{2} \int_{-\infty}^\infty dq H_n^{(1)}(qr)F(q) \quad (17)$$

where according to (15) function  $F(q)$  is odd/even for even/odd values of  $n$ . In (17), we have used the symmetry of the Hankel functions  $H_n^{(2)}(-x) = -e^{in\pi} H_n^{(1)}(x)$ .

Second, we use the asymptotic form of the Hankel functions for large arguments ( $qr \gg 1$ ). Notice that the region that provides the major contribution to the integral corresponds to  $q \geq 1$ . Then, the formal condition of the asymptotic expansion validity for the lower value of the contributing  $q$  reads as  $r \gg 1$ . However, as we will show in the following, the true region of distances where the asymptotic approximation is valid is much less restricted. Retaining the first two terms in  $H_n^{(1)}(qr)$ , the expansion reads

$$H_n^{(1)}(x) = \sqrt{\frac{2}{\pi x}} e^{i[x - \frac{\pi}{2}(n + \frac{1}{2})]} \left( 1 + i \frac{4n^2 - 1}{8x} \right) + O(x^{-\frac{5}{2}}). \quad (18)$$

For convenience, let us normalize the DGF as follows:

$$\hat{G} = \frac{k_\omega e^{i\frac{\pi}{4}}}{8\pi} \sqrt{\frac{2}{\pi r}} \hat{g}. \quad (19)$$

Then, using (16) and (17), we find from (15) the expressions for  $\hat{g} = \hat{g}_s + \hat{g}_p$

$$\hat{g}_\tau(r, z) = \int_{-\infty}^\infty dq \frac{e^{iqr + iq_z z}}{f_\tau(q)} \left[ \hat{A}_\tau^{(1)}(q) + \frac{i}{r} \hat{A}_\tau^{(2)}(q) \right] \quad (20)$$

with  $\tau = s, p$  and the superscripts (1, 2) of  $A$  indicate distinct dependences upon  $r$ . The denominators in the integrand are

$$f_s(q) = \alpha + q_z, \quad f_p(q) = \alpha q_z + 1 \quad (21)$$

and nominators are

$$\begin{aligned} \hat{A}_p^{(1)}(q) &= \sqrt{q} \left[ q_z \hat{r} \hat{r} - q (\hat{r} \hat{z} + \hat{z} \hat{r}) + \frac{q^2}{q_z} \hat{z} \hat{z} \right] \\ \hat{A}_s^{(1)}(q) &= \sqrt{q} \hat{\phi} \hat{\phi} \\ \hat{A}_p^{(2)} &= \frac{1}{8\sqrt{q}} \left[ 7q_z \hat{r} \hat{r} - 8q_z \hat{\phi} \hat{\phi} - 3q (\hat{r} \hat{z} + \hat{z} \hat{r}) - \frac{q^2}{q_z} \hat{z} \hat{z} \right] \\ \hat{A}_s^{(2)} &= \frac{1}{\sqrt{q}} \left( -\hat{r} \hat{r} + \frac{7}{8} \hat{\phi} \hat{\phi} \right) \end{aligned} \quad (22)$$

where  $\hat{r}$ ,  $\hat{\phi}$ , and  $\hat{z}$  represent the unit vectors.

As we see, the integrands contain branch points corresponding to  $q_z$ , and branch cuts corresponding to  $\text{Im}(q_z) = 0$ . In order to remove these problematic peculiarities, we perform the following standard change of integration variable and coordinates [see Fig. 2(b)]<sup>2</sup>:

$$q = \sin \varphi, \quad q_z = \cos \varphi, \quad r = \rho \sin \theta, \quad z = \rho \cos \theta. \quad (23)$$

<sup>2</sup>Notice that with this substitution, the branch cut in the integrand only disappears for the symmetric dielectric surrounding of the conductivity sheet. In the event of different dielectrics, in the substrate and superstrate half-spaces, a branch cut will be encountered in the new integration plane, resulting in an additional asymptotic contribution (as described in [13]).

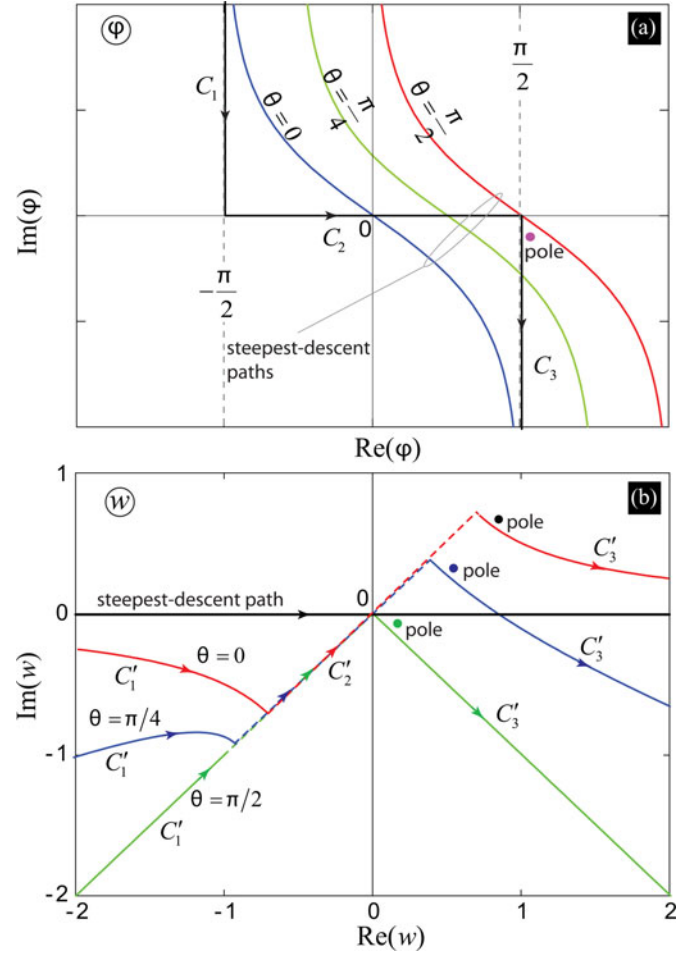


Fig. 3. Integration contours in the complex plane of the integration variables (a)  $\varphi$  and (b)  $w$  for different values of angle  $\theta$ . In panels (a) and (b), the initial integration contour corresponds to  $C = C_1 + C_2 + C_3$  and  $C' = C'_1 + C'_2 + C'_3$ , respectively. The pole position is shown by a circular symbol. In the complex plane of  $\varphi$ , the initial path is the same and the steepest descent path depends upon  $\theta$ , while in the complex plane  $w$ , the steepest descent path is the same,  $\text{Im}(w) = 0$ , and the initial integration path changes with  $\theta$ . The position of the pole in the complex plane  $w$  is dependent upon  $\theta$ .

In the general case, we cannot get rid of substrate/superstrate. The integrals (20) then transform as

$$\begin{aligned} \int_{-\infty}^\infty dq e^{iqr + iq_z z} \frac{\hat{A}_\tau^{(n)}(q)}{f_\tau(q)} \\ = \int_C d\varphi e^{i\rho \cos(\varphi - \theta)} \cos \varphi \frac{\hat{A}_\tau^{(n)}(\sin \varphi)}{f_\tau(\sin \varphi)} \end{aligned} \quad (24)$$

where the integration contour  $C$  passes through the complex plane  $\varphi$  [see Fig. 3(a)] and corresponds to the real axis in the complex  $q$ -plane. At this stage, we can perform a steepest descent integration. Then, the integration path must be transformed to  $\cos[\text{Re}(\varphi) - \theta] \cosh[\text{Im}(\varphi)] = 1$  [see Fig. 3(a)], with the saddle point  $\varphi = \theta$ . However, the steepest descent integration is much easier in another complex variable plane. This variable  $w$  is given as follows:

$$w = \sqrt{2} e^{i\frac{\pi}{4}} \sin \left( \frac{\varphi - \theta}{2} \right) \quad (25)$$



so that  $i \cos(\varphi - \theta) = i - w^2$ . With this change of variable,  $\hat{g}_\tau$  becomes

$$\hat{g}_\tau = e^{i\rho} \int_{C'} dw e^{-\rho w^2} \left[ \hat{\Phi}_\tau^{(1)}(w) + \frac{i}{r} \hat{\Phi}_\tau^{(2)}(w) \right]$$

$$\hat{\Phi}_\tau^{(n)}(w) = \cos[\varphi(w)] \cdot \frac{d\varphi}{dw} \cdot \frac{\hat{A}_\tau^{(n)}[q(w)]}{f_\tau[q(w)]} \quad (26)$$

where the integration contour  $C'$  is shown in Fig. 3(b). The steepest descent integration path is now simply given by  $\text{Im}(w) = 0$ , and the saddle point is located in the origin,  $w = 0$ . Notice that the branch points  $q = \pm 1$  (corresponding to  $q_z = 0$ ) are given by  $\varphi = \pm\pi/2$  in the complex plane  $\varphi$ , while in the  $w$ -plane, they are located at  $w = \sqrt{2}e^{i\frac{\pi}{4}} \sin(\pm\frac{\pi}{4} - \frac{\theta}{2})$ . For  $\theta = \pm\pi/2$ , the branch point and saddle point in  $w$ -plane coincide.

An important point here is that the elements of the integrand dyadic in  $\hat{g}_\tau$  are singular due to the presence of the poles [ $f_p(q) = 0$  and  $f_s(q) = 0$ ] in the denominators. These poles are located at  $q = q_p$  and  $q = q_s$ , respectively. They correspond to TM surface wave (GSP) and TE surface wave [2]

$$q_p = \sqrt{1 - \frac{1}{\alpha^2}}$$

$$q_s = \sqrt{1 - \alpha^2}. \quad (27)$$

The positions of the poles depend essentially upon the value of the normalized conductivity  $\alpha$ . TM waves correspond to  $\text{Im}(\alpha) > 0$ , while TE ones correspond to  $\text{Im}(\alpha) < 0$  so that TM and TE surface waves cannot exist at the same frequency, see [2]. High values of  $|\alpha|$  correspond to large  $q_s$ , while low values of  $|\alpha|$  correspond to large  $q_p$ . In principle, taking into account a wide range of metamaterials that are available at present, a wide range of  $\alpha$  is also accessible. A mathematical treatment of the problem corresponding to a 3-D layer of a very thin thickness  $h \ll \lambda$  with the dielectric permittivity  $\varepsilon_{3D}$  can be reduced to the case of a 2-D sheet with an effective 2-D normalized conductivity  $\alpha_{\text{eff}}$ . The relation between 2-D effective conductivity and 3-D permittivity is established from the comparison of the Fresnel coefficients and reads as  $\alpha_{\text{eff}} = \pi h \varepsilon_{3D} / i\lambda$ .

Returning to the case of graphene, due to small values of  $|\alpha|$ , TE surface waves are very weakly bounded ( $|q_s| \sim 1$ ) and therefore they virtually do not couple to the point emitter. This means that, in practice, the contribution from TE pole can be neglected in graphene [8]. Nevertheless, taking into account a wide range of possible 2-D sheets, in our analysis we retain the contribution from both poles.

In order to proceed with the series expansion, the singular terms in the integrands must be separated. The separation for the dyadics  $\hat{\Phi}_\tau^{(n)}(w)$  into a pole and a smooth part  $\hat{\Phi}_{S\tau}^{(n)}(w)$  is as follows:

$$\hat{\Phi}_\tau^{(n)}(w) = \frac{\hat{Q}_\tau^{(n)}}{w - w_\tau} + \hat{\Phi}_{S\tau}^{(n)}(w)$$

$$\hat{\Phi}_{S\tau}^{(n)}(w) = \frac{\hat{\Phi}_\tau^{(n)}(w)(w - w_\tau) - \hat{Q}_\tau^{(n)}}{w - w_\tau} \quad (28)$$

where  $\hat{Q}_\tau^{(n)}$  are dyadics with the elements corresponding to the residues of  $\hat{\Phi}_\tau^{(n)}(w)$ , which after some algebra can be computed

from (26) as

$$\hat{Q}_p^{(n)} = \frac{1}{\alpha^2 q_p} \hat{A}_p^{(n)}(q_p), \quad \hat{Q}_s^{(n)} = \frac{\alpha}{q_s} \hat{A}_s^{(n)}(q_s). \quad (29)$$

Let us explicitly write out the expressions for the residue dyadics

$$\hat{Q}_p^{(1)} = \frac{\sqrt{q_p}}{\alpha^2} \begin{pmatrix} \frac{-1}{\alpha q_p} & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -\alpha q_p \end{pmatrix}$$

$$\hat{Q}_s^{(1)} = \frac{-\alpha}{\sqrt{q_s}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{Q}_p^{(2)} = \frac{1}{8\alpha^2 \sqrt{q_p}} \begin{pmatrix} \frac{-7}{\alpha q_p} & 0 & 3 \\ 0 & \frac{-8}{\alpha q_p} & 0 \\ 3 & 0 & \alpha \end{pmatrix}$$

$$\hat{Q}_s^{(2)} = \frac{\alpha}{q_s^{\frac{3}{2}}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{-7}{8} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (30)$$

Now, we can deform the integration contour  $C'$  in the  $w$  complex plane into the real  $w$ -axis. Then, the singular terms  $\frac{\hat{Q}_\tau^{(1,2)}}{w - w_\tau}$  in (28) for  $\hat{\Phi}_\tau^{(1,2)}(w)$  that enter to the integral (28) can be integrated analytically. The result for  $\hat{g}_\tau(r, z)$  can be presented in the form of a sum

$$\hat{g}_\tau(r, z) = i\pi e^{\rho(i - w_\tau^2)} \text{erfc}(-i w_\tau \sqrt{\rho})$$

$$\times \left( \hat{Q}_\tau^{(1)} + \frac{i}{r} \hat{Q}_\tau^{(2)} \right) + \hat{g}_{0\tau}(r, z) \quad (31)$$

where the term with the complementary error function  $\text{erfc}(x) = (2/\sqrt{\pi}) \int_x^\infty e^{-t^2} dt$  is due the singularity. This function includes the contribution from the pole that is automatically taken into account when the pole is crossed by the transformation of the integration contour. The second term in (31)  $\hat{g}_{0\tau}(r, z)$  presents a nonsingular contribution

$$\hat{g}_{0\tau}(r, z) = e^{i\rho} \int_{-\infty}^{\infty} dw e^{-\rho w^2} \left[ \hat{\Phi}_{S\tau}^{(1)}(w) + \frac{i}{r} \hat{\Phi}_{S\tau}^{(2)}(w) \right]. \quad (32)$$

The integral appearing in (32) is of the Gauss type, so the functions  $\hat{\Phi}_{S\tau}^{(n)}(w)$  can be expanded in Taylor series close to  $w = 0$  and integration of every term is easily performed with the following result:

$$\hat{g}_{0\tau}(r, z) = e^{i\rho} \sum_{m \in \text{Even}} \frac{1}{m!} \frac{\Gamma(\frac{1+m}{2})}{\rho^{\frac{1+m}{2}}}$$

$$\frac{d^m}{dw^m} \left[ \hat{\Phi}_{S\tau}^{(1)}(w) + \frac{i}{r} \hat{\Phi}_{S\tau}^{(2)}(w) \right]_{w=0} \quad (33)$$

where  $\Gamma$  is Gamma function. Retaining in this expression the terms up to order  $r^{-3/2}$  (which is enough for most of the cases),

we arrive at

$$\hat{g}_{0\tau}(r, z) \simeq e^{i\rho} \sqrt{\frac{\pi}{\rho}} \left[ \hat{\Phi}_{S\tau}^{(1)}(0) + \frac{i}{r} \hat{\Phi}_{S\tau}^{(2)}(0) + \frac{1}{4\rho} \frac{d^2 \hat{\Phi}_{S\tau}^{(1)}}{dw^2} \Big|_{w=0} \right]. \quad (34)$$

Recalling that the saddle point  $w = 0$  corresponds to  $\varphi = \theta$  and therefore to  $q = \sin \theta$ , we can explicitly write out the dyadics  $\hat{\Phi}_{S\tau}^{(n)}(0)$

$$\hat{\Phi}_{S\tau}^{(n)}(0) = \frac{\hat{Q}_\tau^{(n)}}{w_\tau} + \sqrt{2} e^{-i\frac{\pi}{4}} \frac{\hat{A}_\tau^{(n)}(\sin \theta)}{f_\tau(\sin \theta)} \cos \theta \quad (35)$$

where we have taken into account the identity  $\frac{d\varphi}{dw} = \sqrt{2} e^{-i\frac{\pi}{4}} / \cos(\frac{\varphi-\theta}{2})$ . The explicit expression for the third term in (34) is more cumbersome for arbitrary  $\theta$ ; therefore, we give its formal expression, involving derivatives of previously defined functions

$$\begin{aligned} \frac{d^2 \hat{\Phi}_{S\tau}^{(1)}}{dw^2} \Big|_{w=0} &= \frac{2\hat{Q}_\tau^{(1)}}{w_\tau^3} + \frac{d^2 \hat{\Phi}_\tau^{(1)}}{dw^2} \Big|_{w=0} \\ \frac{d^2 \hat{\Phi}_\tau^{(1)}}{dw^2} \Big|_{w=0} &= 2\sqrt{2} e^{-i\frac{3\pi}{4}} \frac{d^2}{d\varphi^2} \left[ \frac{\cos \varphi}{\cos(\frac{\varphi-\theta}{2})} \frac{\hat{A}_\tau^{(1)}(\sin \varphi)}{f_\tau(\sin \varphi)} \right]_{\varphi=\theta}. \end{aligned} \quad (36)$$

Expressions in (31), with  $\hat{g}_{0\tau}(r, z)$  given by (34), present the analytical approximation of the DGF. Let us now analyze different terms in this expression and the applicability of the approximation.

## V. ANALYSIS OF THE ANALYTICAL SOLUTION

In its general form, the analytical solution presents a nontrivial combination of the contribution from the pole and saddle point. The ‘‘interaction’’ between these contributions depends both upon the distance between the saddle point and the pole in the complex plane and upon the physical distance  $r$  responsible for the oscillations of the integrand. The parameter that measures the interaction between the saddle point and the pole is called by Sommerfeld ‘‘numerical distance’’  $d_\tau$  and its square presents the argument of the complementary error function,  $d_\tau^2 = -iw_\tau \sqrt{\rho}$ .

In the  $q$ -plane, the saddle point corresponds to the condition of the extremum of the exponential phase in (20), i.e., to  $(q/q_z)_{\min} = r/z$ , or  $q_{\min} = \sin \theta$ . This can be considered as an equation for the rays in ‘‘ray optics’’ (RO). If the contribution of the pole is neglected, then the same result can be derived following the standard stationary phase evaluation. In the far field, the leading RO contribution corresponds to the first term in (34) with  $\hat{\Phi}_{S\tau}^{(1)}(0)$  replaced by  $\hat{\Phi}_\tau^{(1)}(0)$ . It reads

$$\hat{g}_\tau^{\text{RO}}(r, z) = e^{i\rho - i\frac{\pi}{4}} \sqrt{\frac{2\pi}{\rho}} \frac{\hat{A}_\tau^{(n)}(\sin \theta)}{f_\tau(\sin \theta)} \cos \theta. \quad (37)$$

Returning to the DGF via (19), we have explicitly [using (21), (22), and the relation  $r = \rho \sin \theta$ ]

$$\hat{G}_p^{\text{RO}}(\rho, \theta) = \frac{k_\omega e^{i\rho} \cos \theta}{4\pi\rho(1 + \alpha \cos \theta)} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 0 & 0 \\ -\sin \theta & 0 & \frac{\sin^2 \theta}{\cos \theta} \end{pmatrix}$$

$$\hat{G}_s^{\text{RO}}(\rho, \theta) = \frac{k_\omega e^{i\rho} \cos \theta}{4\pi\rho(\alpha + \cos \theta)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (38)$$

When the graphene sheet disappears,  $\alpha \rightarrow 0$ , we recover the spherical wave term ( $\sim 1/\rho$ ) of DGF corresponding to a dipole in FS. Notice that at  $\theta = \pi/2$  (for the fields along  $z = 0$ ), the elements  $rr, rz$ , and  $zr$  in  $\hat{G}_p^{\text{RO}}$  vanish both when graphene sheet is present and when it is not. In contrast, the element  $\phi\phi$  in  $\hat{G}_s^{\text{RO}}$  at  $\theta = \pm\pi/2$  is very sensitive to the presence of graphene. It takes nonzero values for FS,  $\alpha = 0$  and vanishes for  $\alpha \neq 0$ . This property of the  $\phi\phi$  element is similar to the diffraction shadow effect in metals due to the presence of surface modes and formation of the Norton waves (NW) [23]. In case of metals, however, the diffraction shadow appears for the TM part of the dyadic, while for thin films, this takes place for the TE part.

The fact that some elements of the dyadic  $\hat{G}^{\text{RO}}$  vanish indicates that other terms in the general solution for  $\hat{G}$  must be considered in order to provide the correct far-field representation of the DGF and electric fields. Let us consider in details the case of  $\theta = \pi/2$ .

### A. Asymptotic Expansion of DGF in Graphene Plane, $z = 0^-$ ( $\theta = \pi/2$ )

Here, we present simplified expressions for  $\hat{g}_\tau(r, z)$  at  $z = 0$ . Recall that the general expression is given by (31), with  $\hat{g}_{0\tau}$  given by (34). The functions  $\hat{\Phi}_\tau^{(n)}(0)$  at  $\theta = \pi/2$  read

$$\begin{aligned} \hat{\Phi}_s^{(1)}(0) &= \hat{\Phi}_s^{(2)}(0) = 0, \\ \hat{\Phi}_p^{(1)}(0) &= \sqrt{2} e^{-i\frac{\pi}{4}} \hat{z}\hat{z}, \quad \hat{\Phi}_p^{(2)}(0) = -\frac{\sqrt{2}}{8} e^{-i\frac{\pi}{4}} \hat{z}\hat{z}. \end{aligned} \quad (39)$$

Introducing for a brevity of notations, the following dyadic:

$$\frac{d^2 \hat{\Phi}_s^{(1)}}{dw^2} \Big|_{w=0} \equiv \hat{M}_\tau \quad \text{for } \theta = \frac{\pi}{2} \quad (40)$$

derivatives (36) simplify as

$$\begin{aligned} \hat{M}_p &= 4\sqrt{2} e^{-i\frac{3\pi}{4}} \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & \alpha^2 - \frac{9}{8} \end{pmatrix} \\ \hat{M}_s &= \frac{4\sqrt{2}}{\alpha^2} e^{i\frac{\pi}{4}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (41)$$

Now, we can explicitly write the full expression for the TE dyadic

$$\begin{aligned} \hat{G}_s(r, 0) &= \frac{k_\omega e^{i\frac{3\pi}{4}}}{8\pi} \sqrt{\frac{2\pi}{r}} \left( \hat{Q}_s^{(1)} + \frac{i}{r} \hat{Q}_s^{(2)} \right) e^{iq_s r} \text{erfc}(-iw_s \sqrt{r}) \\ &+ \frac{k_\omega e^{ir+i\frac{\pi}{4}}}{4\pi\sqrt{2}r} \left[ \frac{1}{w_s} \left( \hat{Q}_s^{(1)} + \frac{i}{r} \hat{Q}_s^{(2)} \right) + \frac{1}{4r} \left( \frac{2\hat{Q}_s^{(1)}}{w_s^3} + \hat{M}_s \right) \right] \end{aligned} \quad (42)$$

and for the TM one

$$\hat{G}_p(r, 0) = \frac{k_\omega e^{i\frac{3\pi}{4}}}{8\pi} \sqrt{\frac{2\pi}{r}} \left( \hat{Q}_p^{(1)} + \frac{i}{r} \hat{Q}_p^{(2)} \right) e^{iq_p r} \text{erfc}(-iw_p \sqrt{r})$$

$$\begin{aligned}
 & + \frac{\hat{k}_\omega e^{i\tilde{r} + i\frac{\pi}{4}}}{4\pi\sqrt{2r}} \left[ \frac{1}{w_p} \left( \hat{Q}_p^{(1)} + \frac{i}{r} \hat{Q}_p^{(2)} \right) \right. \\
 & \left. + \sqrt{2} e^{-i\frac{\pi}{4}} \left( 1 + \frac{i}{8r} \right) \hat{z}\hat{z} + \frac{1}{4r} \left( \frac{2\hat{Q}_p^{(1)}}{w_p^3} + \hat{M}_p \right) \right].
 \end{aligned} \quad (43)$$

Recall that in (42) and (43), the residue dyadics  $\hat{Q}_\tau^{(n)}$  are given by (30), and  $\hat{M}_\tau$  have the form of (41). The locations of the poles in the complex  $w$ -plane at  $\theta = \pi/2$  read

$$w_{s,p} = e^{-i\frac{\pi}{4}} \sqrt{q_{s,p} - 1}. \quad (44)$$

We have checked that (42) and (43) transform to the DGF of FS in the limit  $\alpha \rightarrow 0$ . To perform this limit, one should carefully expand all the coefficients taking into account that for small  $\alpha$ , the poles become  $w_s \simeq \frac{\alpha}{\sqrt{2}} e^{i\frac{\pi}{4}}$ ,  $w_p \simeq \frac{i}{\sqrt{\alpha}}$ . In particular, for TM part of the DGF, the expansion for large arguments of the complementary error function must be taken.

### B. Numerical Check on the Validity of the Analytical Approximation

Let us present some illustrative examples that demonstrate the validity of the analytical approximation. For this, we directly compare numerical and analytical calculations for two components of DGF in the most unfavorable situation, i.e., for  $z = 0$  ( $\theta = \pi/2$ ), when the ‘‘RO’’ contribution disappears. We perform the precise (converged) numerical calculations according to Appendix A. The analytical approximation is given by (42) and (43).

The conductivity of graphene is a function of frequency,  $\nu = \omega/(2\pi)$ , chemical potential  $\mu$ , temperature  $T$ , and scattering time  $\tau$  (see Appendix B). For illustration, we consider the room temperature  $T = 300$  K and  $\mu = 0.2$  eV (48 THz), which are typical experimental values.

The comparison is shown in Fig. 4. The range of the distances corresponds to the subwavelength region (outside of this region, the numerical and analytical curves are virtually undistinguishable). In order to characterize the difference between numerical  $\hat{G}^n$  and analytical  $\hat{G}^a$  results, in the insets we have represented the relative error  $\Delta_{\beta\beta'} = \left| \left( G_{\beta\beta'}^n - G_{\beta\beta'}^a \right) / G_{\beta\beta'}^n \right|$  in THz frequency range. According to the insets in Fig. 4, the error is a nonmonotonous function of frequency. However, it has a decaying tendency with frequency increase. To understand such behavior, let us notice that for higher frequencies,  $|\alpha|$  decreases so that both the real and imaginary parts of  $q_p$  increase. In the lower limit ( $\nu = 1$  THz),  $\alpha \simeq 0.11 + 0.69i$  so that  $q_p \simeq 1.7 + 0.19i$ , while in the upper limit ( $\nu = 10$  THz)  $\alpha \simeq 0.0016 + 0.07i$  so that  $q_p \simeq 14.34 + 0.34i$ . Thus, the propagation length of the GSP,  $L_{\text{GSP}} = \lambda/[2\pi\text{Im}(q_p)]$ , decreases due to an increase in  $\text{Im}(q_p)$ . This leads to a strong spatial decay of the terms in the solution (at the deep subwavelength distance), related to the GSP field components. Since the analytical solution recovers the FS DGF (up to  $1/r^2$ ), the coincidence between the analytical and exact solutions improves for higher frequencies.

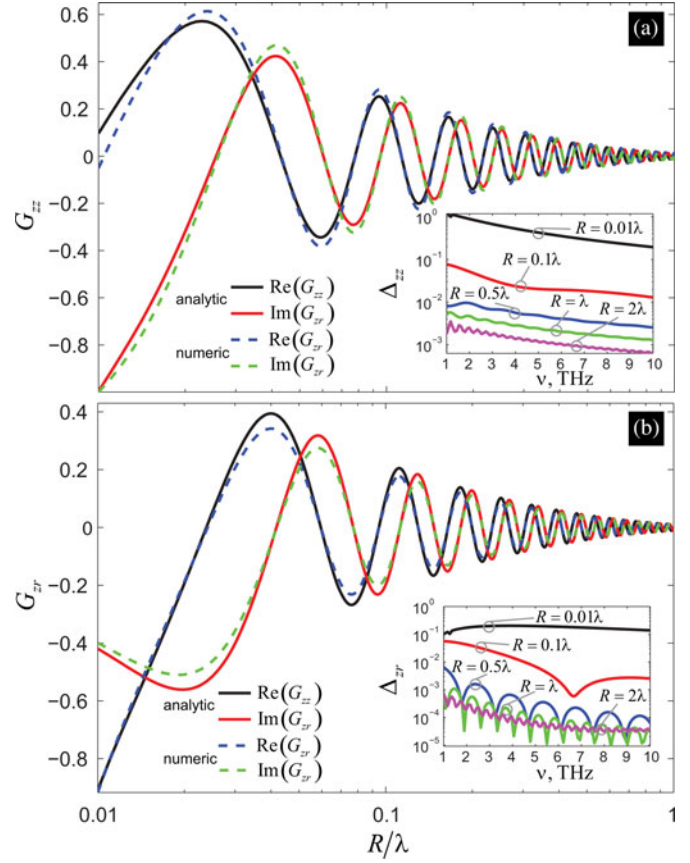


Fig. 4. Comparison between the numeric (see Appendix A) and analytic calculations of  $zz$  and  $zr$  DGF elements at  $z = 0$ . The main figures show both real and imaginary parts of  $G_{zz}$  and  $G_{zr}$  as a function of distance. The values of these elements have been normalized to the maximal values of their modules in the shown range of distances. The parameters for graphene in both panels are:  $T = 300$  K,  $\mu = 0.2$  eV,  $\tau = 1$  ps, and  $\nu = 10$  THz. The insets show the dependences of the relative error upon the frequency for different distances. In the insets,  $\mu$  and  $\tau$  are the same as in main figures.

## VI. LONG-DISTANCE LIMIT. SURFACE MODES AND ALGEBRAICALLY DECAYING COMPONENTS

In the region of parameters, where the argument of the complementary error function is a large number  $|w_\tau| \sqrt{\rho} \gg 1$  (the numerical distance is large,  $|d_\tau^2| \gg 1$ ), this function can be substituted by a few terms from its asymptotic expansion

$$\begin{aligned}
 \text{erfc}(-iw_\tau \sqrt{\rho}) &= 2\Theta_- [\text{Im}(w_\tau)] + \frac{e^{w_\tau^2 \rho}}{w_\tau \sqrt{\pi \rho}} \\
 &\times \sum_{n=0}^{\infty} \frac{(-1)^n}{(-i)^{2n+1}} \frac{(2n)!}{n! (2w_\tau)^{2n} \rho^n} \quad (45)
 \end{aligned}$$

where

$$\begin{aligned}
 \Theta_-(x) &= 0, \quad \text{for } x \geq 0 \\
 \Theta_-(x) &= 1, \quad \text{for } x < 0.
 \end{aligned} \quad (46)$$

The first term (which is independent upon  $\rho$ ) appears when the transformation of the initial integration path to the steepest descent one results in crossing the pole. Retaining the terms

exact up to  $r^{-3/2}$  in (45), we arrive at

$$\operatorname{erfc}(-iw_\tau\sqrt{\rho}) = 2\Theta_- [\operatorname{Im}(w_\tau)] + \frac{ie^{w_\tau^2\rho}}{w_\tau\sqrt{\pi\rho}} \times \left(1 + \frac{1}{2w_\tau^2\rho}\right) + O[(w_\tau\sqrt{\rho})^{-5}]. \quad (47)$$

Let us concentrate on the case of the in-plane fields,  $\theta = \pi/2$ . Taking into account that  $e^{w_\tau^2 r} e^{iq_\tau r} = e^{ir}$ , (42) and (43) simplify to

$$\hat{G}_s(r, 0) = \frac{k_\omega e^{i\frac{\pi}{4}}}{8\pi} \sqrt{\frac{2}{\pi r}} \left[ \hat{\Pi}_s e^{iq_s r} + e^{ir} \frac{\sqrt{\pi}}{4r\sqrt{r}} \hat{M}_s \right] + O\left(\frac{1}{r^3}\right) \quad (48)$$

$$\hat{G}_p(r, 0) = \frac{k_\omega e^{i\frac{\pi}{4}}}{8\pi} \sqrt{\frac{2}{\pi r}} \left[ \hat{\Pi}_p e^{iq_p r} + e^{ir-i\frac{\pi}{4}} \sqrt{\frac{2\pi}{r}} \left(1 + \frac{i}{8r}\right) \hat{z}\hat{z} + e^{ir} \frac{\sqrt{\pi}}{4r\sqrt{r}} \hat{M}_p \right] + O\left(\frac{1}{r^3}\right) \quad (49)$$

where the dyadic  $\hat{\Pi}_\tau$  describes the surface mode and only contributes when the pole is located in the physically proper Riemann sheet

$$\hat{\Pi}_\tau = 2\pi i \cdot \Theta_- [\operatorname{Im}(w_\tau)] \left( \hat{Q}_\tau^{(1)} + \frac{i}{r} \hat{Q}_\tau^{(2)} \right). \quad (50)$$

We would like to notice that in the approximate expression for  $\hat{g}_s(r, 0)$  given by (48), we cannot recover the limit  $\alpha \rightarrow 0$  anymore.

Expressions (49) and (50) present a sum of the surface mode term proportional to  $\hat{\Pi}_\tau$  (GSP in case of  $p$ -polarization) and algebraically decaying terms. The term describing the surface mode can be directly recovered from the angular representation (20), considering only the residue of the pole. The algebraic components, in their turn, can be derived from the same integral (20), considering the contribution from the branch point  $q_z = 0$  (see [8]). Asymptotic expressions (49) and (50) present thus *independent* contributions from the pole and the branch cut in the initial integration plane  $q$ .

The main physical reason of the validity of this approximation is that for sufficiently long distances, only sharp peculiarities on the density of electromagnetic states (DES) contribute. DES is reflected by the integrand in (20). For large  $r$ , the smooth region of DES is progressively canceled out in the integral, which is eventually dominated by the strong (and rapid) contribution from the pole. The contribution of this pole gives the field of the surface mode. Due to losses, the density of states associated with the pole has a finite width, which causes the exponential decrease of the GSP amplitude with distance (characterized by the surface mode propagation length). Then, the contribution to the integral from either kink or square-root singularity ( $\propto 1/q_z$ ) located at  $q_z = 0$  dominates. This takes place since this kind of features cannot be characterized by a typical width in  $q$ -space (these features are infinitely sharp in  $q$ -space), and they are not as strongly suppressed as the pole contribution when integrated with an oscillatory function. The contribution of the kink/square-root singularity yields the algebraic decay of the DGF components with respect to the distance.

A detailed physical description of the algebraically decaying components of the fields from a point source can be found in [8]. Let us recall here the physical meaning of all the algebraically decaying terms in (48) and (50). These terms appear in the dyadics  $\hat{M}_\tau$ , and the element  $zz$  of  $\hat{G}_p$  contains an additional contribution  $\sim 1/r$ . The algebraically decaying components are composed of both FS terms and NW [22]. The FS terms do not depend upon  $\alpha$  and result from the contribution of the branch-cut singularity ( $1/q_z$ ) that yields the dependence  $\sim 1/r$  and a kink that yields  $\sim 1/r^2$ . Notice that while  $zz$  component (of the TM part) contains both the term decaying as  $\sim 1/r$  and  $1/r^2$ , the FS part of the element  $rr$  has only  $1/r^2$  decay. All the rest of the  $1/r^2$  terms that depend upon  $\alpha$  correspond to the NW (compare with the case of metals, [20] and [23]). We would like to notice that as follows from (48) (where  $\alpha$  has been supposed to have a nonzero value) the element  $\phi\phi$  contains only the NW. However, if we carefully perform the limit  $\alpha \rightarrow 0$  in the initial equation (42), recovering the FS DGF, the element  $\phi\phi$  will contain a  $\sim 1/r$  term. This can be explained by the fact that in the DGF given by its angular representation, (20) has a square-root singularity for  $\alpha = 0$ .

Since the main message of this paper is the analytical treatment of DGF for graphene, let us illustrate the validity of the asymptotic expressions (49) and (50) by comparing two elements ( $zr$  and  $zz$ ) of DGF with the numeric solution.

First, the competition between algebraically decaying and GSP terms is shown in Fig. 5(a) and (b). At the beginning of the shown spatial window, both elements of DGF are dominated by the GSP terms. Then, in the region of  $R \sim (7-9)\lambda$  for  $G_{zr}$  and  $R \sim (4-5)\lambda$ , there is a crossover, where the exponentially decaying GSP is overcome by the algebraically decaying field. The algebraic decay for  $G_{zr}$  corresponds only to NW ( $\sim 1/R^2$ ), since the FS contribution is zero for this element. In contrast, the asymptotic behavior for  $G_{zz}$  corresponds to FS with dominating  $\sim 1/R$  term. The NW term is also present in the element  $zz$ , but its contribution is much weaker than that of the FS component. In the region of the crossover, the field possesses a peculiar two-scaled oscillation behavior (corresponding to the wavelength of the GSP and vacuum wavelength). Notice that the amplitude of the field at the crossover is extremely small, so for this instance, the analysis has mainly an academic value.

Second, in Fig. 5(c) and (d), a direct comparison of numeric and asymptotic results is performed in the interference region. As one can see, the asymptotic approximation perfectly captures all the details of the exact result. We have also checked the validity of our asymptotical expressions of all other DGF elements.

## VII. CONCLUSION

We have performed an analytical treatment of DGF for 2-D sheet. In particular, we have tested the analytical expressions on the case of graphene. We have retained all the necessary terms that provide high precision ( $\sim 1\%$ ) down to distances of  $1/10$  wavelengths and reasonable precision ( $\sim 10\%$ ) down to  $1/100$  wavelength.

For the limit of long distances (in units of plasmon wavelengths), we have presented simplified expressions with separated contribution from the pole (plasmonic field) and from the branch point (algebraically decaying field). These expressions



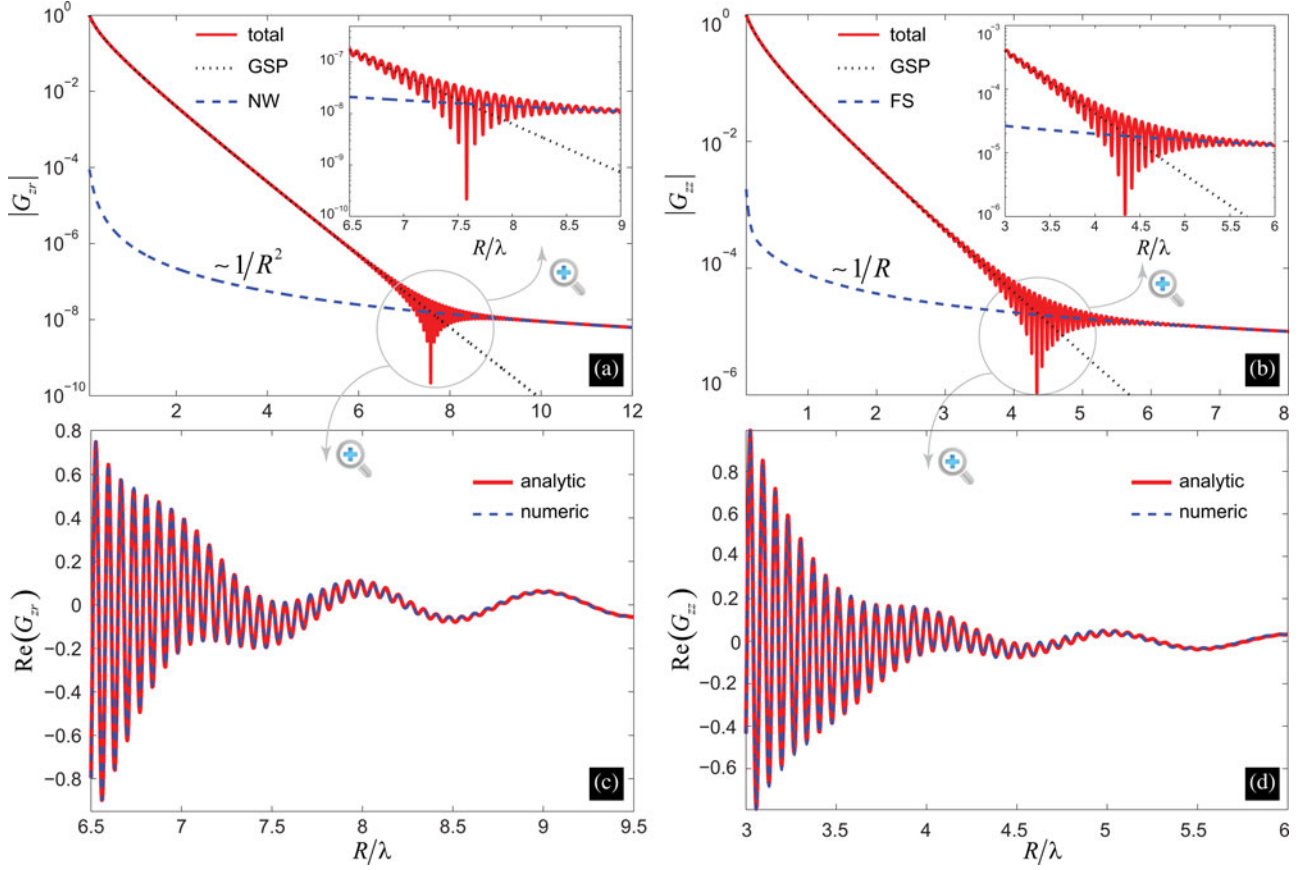


Fig. 5. (a) and (b) Absolute values of  $G_{zr}$  and  $G_{zz}$  as a function of the distance from the point source. The moduli of GSP, NW in (a) and GSP, NW in (b) are also rendered in the same panels. [Inset of (a) and (b)] Zooms of the main panels in the region of a strong interference between GSP and algebraically decaying components (FS and NW). (c) and (d) Real part of  $G_{zr}$  and  $G_{zz}$  as a function of the distance from the point source in the region of interference between GSP and NW or GSP and FS. The analytical expressions based upon (49) are compared with the exact calculation (see Appendix A). In both (a), (b) and (c), (d), all represented values are normalized to the maximal value of  $|G_{zr}|$  ( $|G_{zz}|$ ) in the shown intervals. The parameters of graphene are the same as in Fig. 4.

are relevant for future studies of the electromagnetic properties of subwavelength objects placed on a graphene sheet.

#### APPENDIX A

##### NUMERICAL COMPUTATIONS OF SOMMERFELD INTEGRALS

Let us consider the integral of the following form:

$$I(r) = \int_0^\infty dq \mathcal{F}(q, r), \quad \mathcal{F}(q, r) = F(q) J_n(qr) \quad (51)$$

where  $J_n$  is the  $n$ th-order Bessel function and the function  $F(q)$  remains finite for  $\text{Im}(q) \rightarrow \infty$ . We suppose that the function  $F(q)$  has a pole at  $q = q_p$  and is dependent upon  $q_z = \sqrt{1 - q^2}$  so that it has branch cuts  $\text{Im}(q_z) = 0$ . The pole and the branch cut are not the only difficulties of the integral. In case of the integration along the real axis of the complex  $q$ -plane, the Bessel function has a strong oscillatory behavior for  $q \gg 1$  and the integration is very delicate. When  $r$  increases, the convergence of the integral becomes worse. In order to stay away from the singularity and remain at the same Riemann sheet, the integration path can be deformed according to Cauchy theorem (supposing

that we do not cross the pole)

$$I(r) = \int_A dq F(q) J_n(qr) + \frac{1}{2} \int_B dq F(q) H_n^{(1)}(qr) + \frac{1}{2} \int_C dq F(q) H_n^{(2)}(qr). \quad (52)$$

Here, the contour  $A$  passes below the real axis rounding the pole and the branch cut and then returns toward the real axis at the point  $q = \delta$  with  $\delta > \text{Re}(q_p)$ , moving into  $\text{Im}(q) \rightarrow \infty$  for  $H_n^{(1)}$  term, and to  $\text{Im}(q) \rightarrow -\infty$  for  $H_n^{(2)}$  (see Fig. 6, the paths marked by “2”). The contours “B” and “C” are restricted by the limiting values  $\text{Im}(q) = \pm \Delta$ .

When the pole is far away from the origin,  $|q_p| \gg 1$ , or/and the distance parameter is large,  $r \gg 1$ , it is convenient to bend the contours before the pole, i.e., choose  $\delta < \text{Re}(q_p)$  (see Fig. 6, the paths marked by “1”). In this case, the pole in the second integral of (52) must be taken into account. The contribution of the pole adds the residue term into (52)

$$I(r) = \int_A dq \dots + \frac{1}{2} \int_B dq \dots + \frac{1}{2} \int_C dq \dots + \pi i \cdot \text{Res}(F, q_p) H_n^{(1)}(q_p r). \quad (53)$$

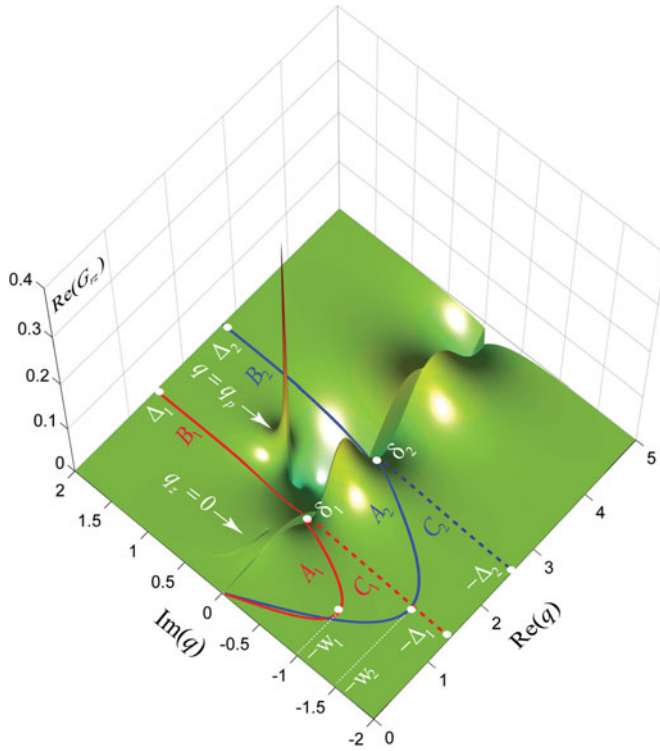


Fig. 6. Contours for the integrand corresponding to the DGF elements  $G_{rz} = G_{zr}$  (the real part of the TM term) for  $z = 0$  and  $r = 0.5$ . The integrand is normalized to the maximal value of its module. In the region  $\text{Im}(q) > 0$  ( $\text{Im}(q) < 0$ ), the Bessel function  $J_1$  is replaced by the decaying Hankel function  $H_1^{(1)}$  ( $H_1^{(2)}$ ). As a result of this replacement, a discontinuity along  $\text{Im}(q) = 0$  appears. The pole position  $q_p = 2 + 0.5i$  corresponds to a “toy value” of the normalized conductivity  $\alpha \simeq 0.168 + 0.516i$ , chosen for better visualization. For the same reasons, in order to better illustrate the branch cut  $\text{Im}(q_z) = 0$ , the causality has been exaggerated as  $\sqrt{1 - q^2} \rightarrow \sqrt{(1 + i0)^2 - q^2} \rightarrow \sqrt{(1 + 0.1i)^2 - q^2}$ . The parameters of the contours are  $\delta_1 = 1.5$ ,  $\delta_2 = 2.75$ ;  $w_1 = 1$ ,  $w_2 = 1.5$ ; and  $\Delta_1 = \Delta_2 = 2$ .

Each path “A”, “B,” and “C” can be parameterized  $q_i = q_i(t)$  ( $i = A, B, C$ ) so that the integration is reduced to the domain  $[0, 1]$

$$I \simeq \sum_i \int_0^1 dt \mathcal{F}[q_i(t)] \frac{dq_i}{dt} \quad (54)$$

with

$$\begin{aligned} q_A(t) &= \delta \cdot t - iw \sin(\pi t) \\ q_B(t) &= \delta + it\Delta \\ q_C(t) &= \delta - it\Delta \end{aligned} \quad (55)$$

where the parameters  $\delta$ ,  $w$ , and  $\Delta$  are chosen so that the best convergence of the integrals is provided. In this paper, we have performed the integration over  $t$  following Simpson’s rule.

## APPENDIX B

### GRAPHENE’S CONDUCTIVITY MODEL

The conductivity of graphene computed within the random phase approximation [16]–[18] can be written through the chemical potential  $\mu$ , the temperature  $T$ , and the scattering energy  $\mathcal{E}_s$

as follows:

$$\sigma = \sigma_{\text{intra}} + \sigma_{\text{inter}} \quad (56)$$

where the intraband and interband contributions are

$$\begin{aligned} \sigma_{\text{intra}} &= \frac{2ie^2 t}{\hbar\pi(\Omega + i\gamma)} \ln \left[ 2 \cosh \left( \frac{1}{2t} \right) \right] \\ \sigma_{\text{inter}} &= \frac{e^2}{4\hbar} \left[ \frac{1}{2} + \frac{1}{\pi} \arctan \left( \frac{\Omega - 2}{2t} \right) \right. \\ &\quad \left. - \frac{i}{2\pi} \ln \frac{(\Omega + 2)^2}{(\Omega - 2)^2 + (2t)^2} \right]. \end{aligned} \quad (57)$$

In these expressions,  $\Omega = \hbar\omega/\mu$ ,  $\gamma = \mathcal{E}_s/\mu$ , and  $t = T/\mu$ , with  $T$  expressed in units of energy. The scattering energy is related to the relaxation time  $\tau$  as  $\tau^{-1} = \mathcal{E}_s/\hbar$ .

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**Alexey Yu Nikitin** received the Ph.D. degree in physics from the Institute of Radiophysics and Electronics, Kharkov, Ukraine, in 2005.

He is currently a Research Fellow at the Instituto de Ciencia de Materiales de Aragon, Consejo Superior de Investigaciones Cientificas, Universidad de Zaragoza, Zaragoza, Spain.

**Francisco J. Garcia-Vidal** received the Ph.D. degree from the Universidad Autónoma de Madrid, Madrid, Spain, in 1992.

Since 1994, he has been working in the field of plasmonics. He is currently a Full Professor in the Departamento de Física Teórica de la Materia Condensada, Universidad Autónoma de Madrid.

**Luis Martin-Moreno** received the Ph.D. degree in physics from the Universidad Autónoma de Madrid, Madrid, Spain, in 1989.

He is currently a Professor at the Instituto de Ciencia de Materiales de Aragon, Consejo Superior de Investigaciones Cientificas, Universidad de Zaragoza, Zaragoza, Spain. He leads a research group working on different theoretical aspects of nanophotonics. From 1989 to 1992, he was a Postdoctoral Research Fellow at the Cavendish Laboratory, Cambridge, U.K. and from 1993 to 1995 he was a Postdoctoral Researcher at the Instituto de Ciencia de Materiales de Madrid, spending several periods at Imperial College London. From then till 2008, he was a Lecturer at the Universidad de Zaragoza.