Theory of Resonant Acoustic Transmission through Subwavelength Apertures

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A complete landscape is presented of the acoustic transmission properties of subwavelength apertures (slits and holes). First, we study the emergence of Fabry-Perot resonances in single apertures. When these apertures are placed in a periodic fashion, a new type of transmission resonance appears in the spectrum. We demonstrate that this resonance stems from the excitation of an acoustic guided wave that runs along the plate, which hybridizes strongly with the Fabry-Perot resonances associated with waveguide modes in single apertures. A detailed discussion of the similarities and differences with the electromagnetic case is also given.

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The properties of periodically distributed acoustic scatterers are receiving growing attention within the physics community. The so-called "phononic crystals" [1–5] or the acoustic metamaterials [6] are just two examples of this type of structures. When exploring the transferability of the phenomenon of extraordinary optical transmission (EOT) [7] to the acoustic case, several studies have also focused on the acoustic properties of periodic arrays of subwavelength apertures [8–12]. Although there are some similarities between the electromagnetic (EM) and acoustic cases, the nonexistence of a cutoff wavelength for acoustic modes propagating through a subwavelength aperture puts some caveats when elaborating the analogy. Moreover, to use the results in acoustics to reveal the underlying physics of EOT, as done in Ref. [11], might lead to wrong conclusions.

In this Letter we present a complete study of the acoustic transmission properties of subwavelength apertures. By using a theoretical framework ideally suited to deal with subwavelength indentations, we start considering the case of a single two-dimensional (2D) square hole and a single one-dimensional (1D) slit perforated on a plate made of a perfect rigid body. Cavity resonances dominate the transmission spectrum and analytical expressions for the transmittance at resonance are given. Next, with the same theoretical formalism, we underpin the physics behind the new transmission resonances appearing in the case of periodic arrays of holes or slits.

The theoretical formalism used throughout this Letter is based on a modal expansion of the pressure field in the different regions defining the structure. We expand the scalar field in a supercell of area $L \times L$ in the xy plane for the case of holes or in a supercell of length L for the case of slits. This supercell is real when we analyze an infinite periodic array or artificial if we study a single aperture. When the structure is finite, the limit $L \rightarrow \infty$ must be taken, leading to analytical expressions for the magnitudes governing the acoustic coupling within the structure. Plane wave expansions are used in the reflection and transmission regions and the direction of propagation of the incident plane wave is chosen to be along the z axis. From now on we denote a plane wave characterized by an in-plane momentum $\vec{k}$ as $|\vec{k}\rangle$. Inside the apertures, the field is written as a linear combination of their eigenmodes (labeled as $|\alpha\rangle$). For the case of square holes of side a, these eigenmodes have analytical expressions and are characterized by two indexes, l and m with $l \geq 0$ and $m \geq 0$. The z component of the wave vector of mode α is expressed as $q_z^\alpha = \sqrt{k_0^2 - (l^2 + m^2)\pi^2/a^2}$, where $k_0 = 2\pi/\lambda$ is the wave number. The important point to realize is that there is no cutoff wavelength for acoustic waves propagating through a 2D hole. When treating 1D slits and $k_y = 0$, only one index (l) characterizes the mode with $l \geq 0$ (no cutoff). As a function of these eigenmodes, the pressure field (p) is expanded within the aperture as

$$p(\mathbf{r}, z) = \sum_\alpha (A_\alpha e^{iq_z^\alpha z} + B_\alpha e^{-iq_z^\alpha z})\langle \mathbf{r}|\alpha\rangle \tag{1}$$

where $\langle \mathbf{r}|\alpha\rangle$ is the in-plane wave field associated with mode α. Note that, as we are assuming that the material forming the perforated plate is a perfect rigid body, the pressure field outside the aperture(s) is zero. This perfect rigid approximation is an excellent starting point when treating stiff materials like steel or brass, materials used in the experiments of Refs. [10,11]. The expansion coefficients in the linear expansion of Eq. (1) are determined after applying the boundary conditions at the two interfaces ($z = 0$ and $z = h$, h being the plate thickness). It is appropriate to define the quantities $v_\alpha = A_\alpha - B_\alpha$ and $v'_\alpha = -(A_\alpha e^{iq_z^\alpha h} - B_\alpha e^{-iq_z^\alpha h})$, which are related to the α-modal amplitudes of the z component of the velocity field at both sides of the aperture. The system of linear equations for the expansion coefficients $[v_\alpha, v'_\alpha]$ is written as

$$\begin{aligned} (G_{\alpha\alpha} - \Sigma_\alpha)v_\alpha + \sum_{\beta\neq\alpha} G_{\alpha\beta}v_\beta - G_\alpha^V v'_\alpha &= I_\alpha, \\ (G_{\alpha\alpha} - \Sigma_\alpha)v'_\alpha + \sum_{\beta\neq\alpha} G_{\alpha\beta}v'_\beta - G_\alpha^V v_\alpha &= 0, \end{aligned} \tag{2}$$

where the analytical expressions for different terms appearing in Eq. (2) are the following. The independent term, I_α depends on the overlap integral between mode $|\alpha\rangle$ and the incident acoustic plane wave, $|\vec{k}_0\rangle$, $I_\alpha = 2i\langle\alpha|\vec{k}_0\rangle$. The coupling between both sides of the aperture is mediated by $G_\alpha^V = q_z^\alpha/[k_0 \sin q_z^\alpha h]$ whereas $\Sigma_\alpha = q_z^\alpha \cos q_z^\alpha h/[k_0 \sin q_z^\alpha h]$ takes into account the multiple rebounds of the acoustic wave inside the aperture. The propagator $G_{\alpha\beta}$ measures the coupling between modes α and β via the diffracted plane waves: $G_{\alpha\beta} = \sum_{\vec{k}} W_{\vec{k}} \langle\alpha|\vec{k}\rangle\langle\vec{k}|\beta\rangle$, where $W_{\vec{k}} = k_0/k_z(\vec{k})$, $k_z = \sqrt{k_0^2 - |k|^2}$, being the z component of the plane wave of in-plane momentum $\vec{k}$. Once the different modal amplitudes are calculated, the transmittance through the structure T is obtained: $T = \sum_\alpha \mathrm{Im}(G_\alpha^V v_\alpha^* v'_\alpha)$.

First, we analyze the case of a single aperture perforated on a plate of thickness h. We have chosen to study a square hole of side $a = 1.06$ mm, which presents the same area as the circular holes forming the 2D array studied in Ref. [10]. In the inset of Fig. 1(a), a contour plot of the normalized-to-area transmittance (normalized to the acoustic energy flux impinging directly at the hole opening) for a normally incident plane wave is shown as a function of λ and h. A set of resonances emerges in the transmission spectrum whose peak wavelengths depend linearly with h, suggesting a Fabry-Perot type origin. Although the existence of these resonances was reported many years ago for circular [13] and rectangular [14] holes, little attention has been paid to analyze in detail their physical origin. When dealing with subwavelength apertures, a very good approximation for the total transmittance is obtained when considering only the first eigenmode ($\alpha \equiv 0$) inside the hole (slit). For this mode, the wave field is constant in the xy plane. In this case, the system of Eqs. (2) becomes two coupled equations:

$$(G - \Sigma)v - G^V v' = I_0, \qquad (G - \Sigma)v' - G^V v = 0, \tag{3}$$

where $\Sigma \equiv \Sigma_0 = \cos k_0 h/\sin k_0 h$ and $G^V \equiv G_0^V = 1/\sin k_0 h$ are the same for both a single hole and a single slit. The main difference between the transmission properties of 1D and 2D single apertures originates from the acoustic coupling between the fundamental eigenmode and all the diffractive waves, i.e., the term $G \equiv G_{00}$ in Eqs. (3). For a square hole of side a, this term can be written explicitly as

$$G_{\mathrm{hole}} = \frac{ia^2 k_0}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk_x dk_y \frac{\mathrm{sinc}^2(\frac{k_x a}{2})\mathrm{sinc}^2(\frac{k_y a}{2})}{\sqrt{k_0^2 - k_x^2 - k_y^2}} \tag{4}$$

with $\mathrm{sinc}(x) = \sin x/x$. A similar expression is obtained for a 1D slit of width a:

$$G_{\mathrm{slit}} = \frac{iak_0}{2\pi} \int_{-\infty}^{+\infty} dk_x \frac{\mathrm{sinc}^2(\frac{k_x a}{2})}{\sqrt{k_0^2 - k_x^2}}. \tag{5}$$

As seen in Fig. 1, the spectral locations of the transmission peaks are associated with the condition $|v| = |v'|$ (i.e., $|G - \Sigma| = |G^V|$) in Eqs. (3). After some algebra, this condition can be written as

$$\tan k_0 h = \frac{2\mathrm{Re}G}{|G|^2 - 1}. \tag{6}$$

In the limit of extremely small apertures ($G \to 0$), this last equation predicts the appearance of transmission peaks close to the condition $\sin k_0 h = 0$. It is even possible to extract some analytical expressions for the normalized-to-area transmittance at resonance (T_{res}) for the case of a single hole or a single slit. By incorporating the resonant condition [Eq. (6)] into the equation for T, it is found that $T_{\mathrm{res}} = |I_0|^2/[4\mathrm{Im}(G)]$. Analytical expressions for T_{res} can be obtained by taking the limit $a \ll \lambda$ of $\mathrm{Im}(G)$ as given in Eqs. (4) and (5). This leads to $T_{\mathrm{res}}^{\mathrm{hole}} = \frac{\lambda^2}{2\pi a^2}$ and $T_{\mathrm{res}}^{\mathrm{slit}} = \frac{\lambda}{\pi a}$. This is a very interesting finding, as it implies an increase of T_{res} as the resonant wavelength is increased. In the EM case, there are no cavity transmission resonances for subwavelength holes due to the existence of a cutoff wavelength. However, this is not the case for a single slit and p-polarized light. Fabry-Perot EM transmission resonances similar to the ones found in the acoustic case have been already reported [15,16].

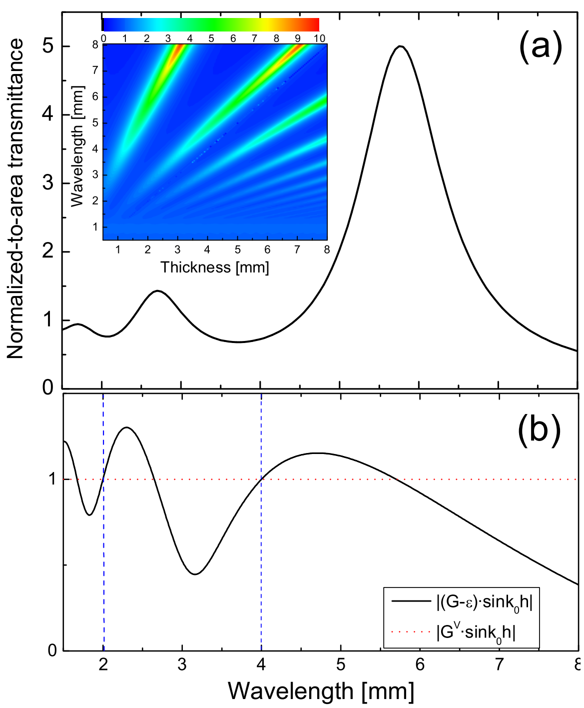

FIG. 1 (color online). (a) Normalized-to-area transmittance (T) spectrum for a normally incident plane wave impinging at a square hole of side $a = 1.06$ mm. The thickness of the plate is $h = 2$ mm. Inset: contour plot of T versus λ and h for a square hole with the same a. (b) $|(G - \epsilon)\sin k_0 h|$ and $|G^V \sin k_0 h|$ versus λ for the case analyzed in panel (a). Vertical dashed lines mark the condition $\sin k_0 h = 0$.

Now we analyze the case of periodic arrays of holes or slits, which can be still analyzed with a system of equations like Eqs. (2). The only change is that now $G_{\alpha\beta}$ needs to be calculated by assuming that only a discrete number of diffracted waves can be excited. In Fig. 2(a), transmittance (in this case normalized to the flux impinging at one unit cell of the structure) for a normally incident plane wave is shown as a function of λ and h. The side of the square holes is like in Fig. 1 ($a = 1.06$ mm) whereas the period $L = 2$ mm, as in the experiments reported in Ref. [10]. Transmission resonances (leading to 100% transmission) whose wavelengths scale linearly with h also appear in the spectrum. However, for wavelengths close to the period of the array, transmittance features are very different to the ones previously found in single holes. For $\lambda = L$, transmittance is zero and, depending on the thickness of the plate, an extremely narrow transmission peak (also reaching 100%) might emerge at a wavelength slightly larger than L [see Fig. 2(b)]. This behavior can be understood by

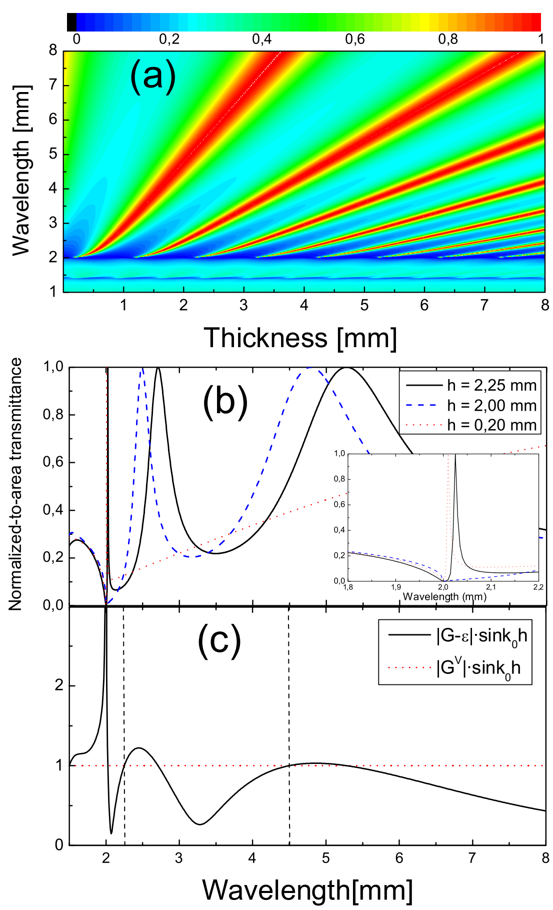

FIG. 2 (color online). (a) Normalized-to-unit-cell transmittance for a normally incident plane wave impinging at a 2D array of square holes of side $a = 1.06$ mm and period $L = 2$ mm as a function of λ and h. (b) The same magnitude for three particular h's: $h = 0.2$, 2, and 2.25 mm as a function of λ. The inset presents a zoom of the three spectra near $\lambda = 2$ mm. (c) $|(G - \epsilon)\sin k_0 h|$ and $|G^V \sin k_0 h|$ versus λ for $h = 2.25$ mm. Vertical dashed lines mark the condition $\sin k_0 h = 0$.

looking at the simpler system of Eqs. (3) for the case of periodic arrays where G is now given by

$$G = i\frac{a^2}{L^2}\sum_n\sum_p \frac{k_0}{k_z^{(n,p)}}\mathrm{sinc}^2\left(\frac{k_x^n a}{2}\right)\mathrm{sinc}^2\left(\frac{k_y^p a}{2}\right), \quad (7)$$

where $k_x^n = k_{0,x} + \frac{2\pi}{L}n$, $k_y^p = k_{0,y} + \frac{2\pi}{L}p$, and $k_z^{(n,p)} = \sqrt{k_0^2 - (k_x^n)^2 - (k_y^p)^2}$. At normal incidence, for $\lambda = L$, $k_z = 0$ for $n = \pm 1$ or $p = \pm 1$ leading to a divergence in G [see Fig. 2(c)] and, consequently, to zero acoustic transmission at that particular λ. Because of the rapid variation of G with λ close to that divergence, for some h's, there is a cut between $|G - \epsilon|$ and $|G^V|$ occurring at a wavelength slightly larger than L [see the case $h = 2.25$ mm in Fig. 2(c)] and leading to an extremely narrow peak in the transmission spectrum. Note that the appearance of this peak is very sensitive to the plate thickness. This sensitivity demonstrates the complex interplay between the Fabry-Perot resonances and the resonant features appearing close to $\lambda = L$ in hole arrays. The transmission properties of a 1D periodic array of slits are analogous to the case of a 2D array of holes. This is due to the fact that both a single slit and a single hole present no cutoff. In Fig. 3(a), normalized-to-unit cell transmission through an array of slits of width $a = 0.5$ mm, period $L = 4.5$ mm, and thicknesses $h = 2$, 4, and 8 mm is plotted as a function of λ. The geometrical parameters correspond to the ones used in the experiments of Ref. [11]. As expected, extremely narrow peaks spectrally located near L also emerge in the spectrum. In Ref. [11], this type of resonances were attributed to the coupling between the composition of diffracted waves and the resonant Fabry-Perot modes inside the apertures. In what follows, we demonstrate that the underlying physics of these resonances is better explained in terms of the excitation of a leaky acoustic guided mode.

Although the surface of a perfect rigid body presents no surface modes, when it is perforated with a periodic array of indentations, surface modes are built up [17]. If the plate is drilled with a 1D array of slits, these acoustic surface modes are always strongly coupled via the waveguide modes in the slits. The result is a mode which is guided along the plate and decays outside it, i.e., an acoustic guided mode. Our theoretical formalism can be used to calculate the dispersion relation (frequency versus k_x) of these acoustic guided modes. In the case of subwavelength slits, the system of Eqs. (3) is still valid but now the system is driven by an evanescent wave of momentum k_x larger than ω/c_s, c_s being the velocity of sound. In this particular case, by neglecting diffraction effects, G in Eqs. (3) is a real magnitude, $G = \frac{a}{L}\frac{k_0}{\sqrt{k_x^2 - k_0^2}}$, and the denominator in Eqs. (3) can be exactly zero at the condition

$$\frac{\sqrt{k_x^2 - k_0^2}}{k_0} = \frac{a}{L}\frac{\sin k_0 h}{\cos k_0 h \pm 1}, \quad (8)$$

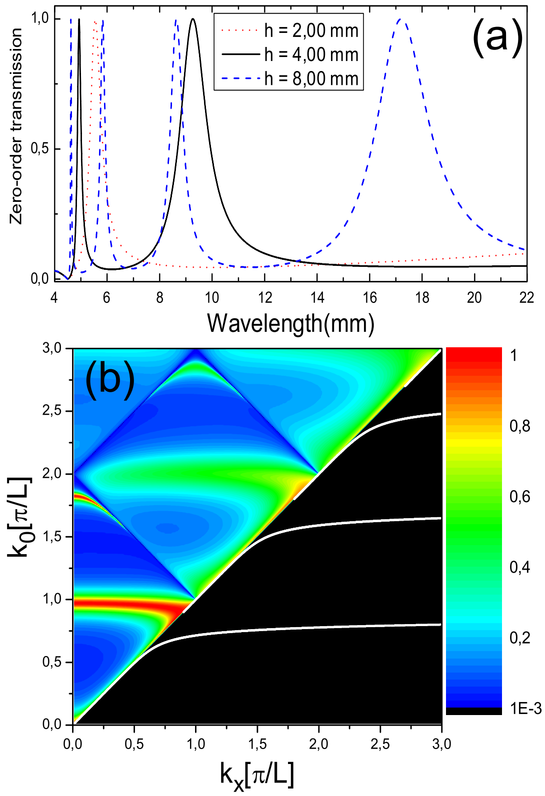

FIG. 3 (color online). (a) Normalized-to-unit-cell transmittance spectra for a normally incident plane wave impinging at an array of slits of width $a = 0.5$ mm and period $L = 4.5$ mm. Three different thicknesses are analyzed $h = 2$, 4, and 8 mm. (b) Left part: the same magnitude as a function of wave number and k_x for the case $h = 4$ mm. Right part: dispersion relation of the acoustic guided modes as obtained from Eq. (8).

where the sign $(+)$ must be taken when $\sin k_0 h > 0$ and sign $(-)$ when $\sin k_0 h < 0$. Equation (8) gives the dispersion relation of the acoustic guided modes for a 1D periodic array of slits in the effective medium limit ($\lambda \gg L, a$). In Fig. 3(b) we show this dispersion relation (white lines in the right part of the panel) for the geometrical parameters of the structure analyzed in Fig. 3(a) with $h = 4$ mm. Two regimes are clearly distinguishable: a linear part close to the "sound line" ($\omega = c_s k_x$) and flat parts that are associated with the Fabry-Perot cavity resonances ($\sin k_0 h = 0$). The important point to realize is that if $\lambda < 2L$, these guided modes become leaky and can be excited by an impinging propagating plane wave. The connection between these leaky guided modes and the transmittance peaks is highlighted in Fig. 3(b). In the left part of this panel transmittance versus wave number and k_x within the sound cone is rendered. It is clear that the location of transmittance peaks can be extracted by just folding the guided modes bands inside the sound cone. We have checked that for the case of 2D hole arrays, this connection between transmission peaks and acoustic guided modes also holds.

Some remarks on the similarities and differences with respect to the EM case are pertinent here. The existence of geometry-induced surface EM modes in 2D hole arrays perforated on perfect conductor plates has been reported recently [18]. These surface EM modes are at the origin of the EOT phenomenon in metals at the THz or microwave regimes. These EM modes have some similarities with the guided modes discussed in this Letter for acoustic waves. However, there is a fundamental difference: in the acoustic case, these modes are not truly surface modes as the two surfaces of the plate are always connected via a propagating wave. This fact provokes that acoustic guided modes always hybridize strongly with the Fabry-Perot resonances associated with the hole or slit cavities.

In conclusion, we have presented a detailed study of the acoustic transmission properties of subwavelength apertures (isolated or forming a periodic array). We expect that this complete study will serve as a well-founded basis in the transferability of the large amount of phenomena found for electromagnetic waves to the acoustic case.

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Note added in proof.—During the review process, the authors were aware of another work [19], in which similar conclusions regarding the transmission properties of 2D hole arrays are reached.

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