*Answers for the Problems on "**Superfluidity in Ultracold Fermi Gases**"*
*(1 tutorial class at "Physics by the Lake")*

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# 1 The ideal Fermi gas

In the LDA, the Fermi distribution function is given by

$$f_F(\mathbf{r},\mathbf{k}) = \frac{1}{\exp[\beta(\epsilon_{\mathbf{k}} + V(\mathbf{r}) - \mu)] + 1} , \tag{1}$$

where $\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m$, $V(\mathbf{r})$ is the harmonic trapping potential. As the chemical potential $\mu$ is fixed by the number of particle equation:

$$N = \int d\mathbf{r} n(\mathbf{r}) = \int d\mathbf{r} \int \frac{d\mathbf{k}}{(2\pi)^3} f_F(\mathbf{r},\mathbf{k}) , \tag{2}$$

one can obtain the expression of the Fermi energy from $\varepsilon_F = \mu(T = 0)$ and from

$$\int_0^{\varepsilon_F} d\epsilon \sqrt{\epsilon}\, (\varepsilon_F - \epsilon)^{3/2} = \frac{\pi}{16} \varepsilon_F^3 .$$

# 2 BCS theory

Remember the properties of the fermionic operators

$$\{c_{\mathbf{k}\sigma}, c^\dagger_{\mathbf{k}'\sigma'}\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'} \qquad \{c^\dagger_{\mathbf{k}\sigma}, c^\dagger_{\mathbf{k}'\sigma'}\} = 0 \qquad \{c_{\mathbf{k}\sigma}, c_{\mathbf{k}'\sigma'}\} = 0 . \tag{3}$$

# 3 BEC-BCS crossover

(a) If we introduce the composite operator

$$b^\dagger_{\mathbf{q}} = \sum_{\mathbf{k}} \varphi_{\mathbf{k}} c^\dagger_{\mathbf{k}+\mathbf{q}/2\uparrow} c^\dagger_{-\mathbf{k}+\mathbf{q}/2\downarrow} ,$$

then, defining $\varphi_{\mathbf{k}} = \tan\theta_{\mathbf{k}}$, we can rewrite the state $|\psi\rangle$ as

$$|\psi\rangle \propto \prod_{\mathbf{k}} e^{\varphi_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}} |0\rangle = e^{b^\dagger_{\mathbf{q}=0}} |0\rangle .$$

This state is a coherent state describing the condensation of the composite bosonic particles $b^\dagger_{\mathbf{q}}$ in the ground state $\mathbf{q} = 0$. Note that this a good description for BEC only at zero temperature, while at finite temperature, the inclusion of thermal fluctuations for pairs (i.e., finite values of $\mathbf{q}$) is essential (see Sec. 2.3.3).

(b) It is useful to introduce the DoS per unit volume at the Fermi surface (see Eqs. (1.11) and (1.17))

$$\mathcal{N}(\varepsilon_F) = \frac{1}{V} N(\varepsilon_F) = \frac{m^{3/2}\sqrt{\varepsilon_F}}{\sqrt{2}\pi^2} = \frac{3}{4}\frac{n}{\varepsilon_F} , \tag{4}$$

in terms of which the scattering length reads $m/4\pi a = (\pi/2)\mathcal{N}(\varepsilon_F)/k_F a$. [1] Introducing the dimensionless units of energy, $x = \epsilon/|\mu|$, [2] one can write the gap equation in the form:

$$\frac{\pi}{2k_F a}\mathcal{N}(\varepsilon_F) = \mathcal{N}(|\mu|)\int_0^\infty dx\sqrt{x}\left[\frac{1}{2x} - \frac{1}{2\sqrt{(x\mp 1)^2 + (\Delta/|\mu|)^2}}\right] ,$$

where the sign $\mp$ corresponds respectively to the cases $\mu >< 0$.

In the BCS limit, we know from the number equation (2.14) that, as $\Delta \ll \varepsilon_F$ (weak coupling regime), the chemical potential $\mu$ differs from the Fermi energy $\varepsilon_F$ by an amount $O(\Delta^2/\varepsilon_F^2)$, $\mu \simeq \varepsilon_F$. Therefore the gap equation now reads

$$\frac{\pi}{2k_F a} = \underbrace{\int_0^\infty dx\sqrt{x}\left[\frac{1}{2x} - \frac{1}{2\sqrt{(x-1)^2 + (\Delta/\varepsilon_F)^2}}\right]}_{\ln(e^2\Delta/8\varepsilon_F)} ,$$

from which one gets the expression (2.19).

In the BEC limit instead $\mu < 0$ and we have seen that $\theta_{\mathbf{k}} \simeq \Delta/2\xi_{\mathbf{k}} \ll 1$ (density of particles in the state $\mathbf{k}$), therefore in the first approximation we have that:

$$\frac{\pi}{2k_F a}\mathcal{N}(\varepsilon_F) \simeq \mathcal{N}(|\mu|)\underbrace{\int_0^\infty dx\sqrt{x}\left[\frac{1}{2x} - \frac{1}{2(x+1)}\right]}_{\pi/2} ,$$

from which we get that $\sqrt{|\mu|/\varepsilon_F} = 1/k_F a$ and therefore Eq. (2.15). One can double-check that the energy scales are arranged as $\varepsilon_F \ll \Delta \ll |\mu|$ by solving the number equation in this limit

$$n \simeq \frac{\mathcal{N}(|\mu|)\Delta^2}{2|\mu|}\underbrace{\int_0^\infty dx\frac{\sqrt{x}}{(x+1)^2}}_{\pi/2} ,$$

which gives $\Delta \simeq \sqrt{16/3\pi}\varepsilon_F\sqrt{1/k_F a}$. [3]

# 4 Magnetised superconductor

Starting from the expression (3.6), consider the gap equation, $\partial f/\partial\Delta = 0$ and then use the BCS result (1.31) for $\Delta_{\mathrm{BCS}}$.

[1] Note that for a single component Fermi gas $\mathcal{N}(\varepsilon_F) = 3n/2\varepsilon_F$.

[2] One has:

$$\frac{1}{V}\sum_{\mathbf{k}} = \int\frac{d\mathbf{k}}{(2\pi)^3} = \int_0^\infty d\epsilon\mathcal{N}(\epsilon) .$$

[3] One can show that expanding to the next order

$$\left(\frac{\Delta}{\varepsilon_F}\right)^2 \simeq \frac{16}{3\pi}\frac{1}{k_F a} + \frac{4}{3\pi^2}(k_F a)^2 .$$