Exam 22/01/2018 — Métodos exp. y comp. de Biofísica (ME)

January 22, 2018

Please send your your script `your_name_exam-ME.m` and external functions at the e-mail address `francesca.marchetti@uam.es`.

Exercise 1

Consider the following function

$$f(x) = \frac{1 - \pi x \cot(\pi x)}{2x^2} ,$$

where $\cot(y) = \frac{1}{\tan(y)}$.

1. Evaluate analytically the first two terms of the Taylor expansion of $f(x)$ around $x = 0$; demonstrate that

$$\lim_{x\to 0} f(x) = \frac{\pi^2}{6} .$$

One can show that the same function is the result of an infinite series:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n^2 + 2n + 1 - x^2} ,$$

2. Evaluate numerically the finite sum

$$a(N) = \sum_{n=0}^{N} \frac{1}{n^2 + 2n + 1} ,$$

for different values of N. Plot $a(N)$ as a function of N and as a function of $\log(N^{-1})$ and check that, in the asymptotic limit, $\lim_{N\to\infty} a(N) = \frac{\pi^2}{6} = f(x \to 0)$.

3. Evaluate the sum numerically for different values of $x \in (-1, 1)$, plot the numerical result by comparing it with the analytical expression of $f(x)$.

Exercise 2

Evaluate numerically the following indefinite integral

$$g(x) = \int_0^x ds \frac{\sin(2s)}{s(s^2+3)} . \tag{1}$$

1. Plot the result in the interval $x \in [0, 2\pi]$.
2. Show that
$$\lim_{x\to+\infty} g(x) = \frac{\pi}{6}\left(1 - e^{-2\sqrt{3}}\right) .$$

Exercise 3: Resonance in a driven damped harmonic oscillator.

In this exercise, we want to demonstrate numerically the resonance phenomenon of a damped harmonic oscillator externally driven by a periodic force. The equation of motion of a one-dimensional driven damped harmonic oscillator is given by

$$\frac{d^2x(t)}{dt^2} + 2\beta\frac{dx(t)}{dt} + \omega_0^2 x(t) = \frac{F(t)}{m} ,$$

and we consider the case of an external sinusoidal drive, $F(t) = F_0 \sin(\omega t)$.

1. Solve numerically this equation for $\omega_0 = 1$ s^{-1}, $\beta = 0.5$ s^{-1}, $F_0/m = 0.2$ m s^{-2}, fixed arbitrary initial conditions $x(t = 0) = x_0$ and $v(t = 0) = v_0$, and drive frequency ω.
2. Demonstrate that the solution $x(t)$ has the following long-time behaviour
$$x(t) \underset{t\gg\beta^{-1}}{\simeq} A(\omega)\cos[\omega t - \delta(\omega)] ,$$
where amplitude and phase shift are given by:
$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega^2-\omega_0^2)^2 + 4\beta^2\omega^2}} \qquad \delta(\omega) = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) .$$
In particular, plot the numerical result obtained for $A(\omega)$ and compare it with the analytical one.

Exercise 4: the symmetric spinning top

It can be shown that, under certain conditions, the rescaled dimensionless total energy E of a symmetric spinning top subject to gravity can be expressed in terms of the "nutation" angle θ and it's angular velocity $\dot{\theta} = \frac{d\theta}{dt}$ (t is here a rescaled dimensionless time) as:

$$E = \frac{1}{2}\dot{\theta}^2 + U_{\text{eff}}(\theta)$$
$$U_{\text{eff}}(\theta) = 1 + 2a\frac{(1-\cos\theta)^2}{\sin^2\theta} + \cos\theta .$$

Here a is a dimensionless parameter to which we will give different values. We are interested in finding the system phase diagram starting from the energy conservation equation and compare these orbits with those obtained by solving the equation of motion directly:

$$\ddot{\theta} = -\frac{dU_{\text{eff}}(\theta)}{d\theta} \ ,$$

where $\ddot{\theta} = \frac{d^2\theta}{dt^2}$.

1. Considering $\theta \in [-\pi, \pi]$, show that the effective potential U_{eff} has a different behaviour for $a \geq 1$ and $a < 1$, evaluate the local extrema in both cases and compare the numerical results with the analytical ones.

2. Find the system phase diagram for the two specific cases a) $a = 2$ and for b) $a = 0.5$ — you have to find the orbits in the plane $\{\theta, \dot{\theta}\}$ for different values of the energy E, i.e., different initial conditions.

3. For $a = 2$ solve numerically the equation of motion for a given set of initial conditions, $\theta(t = 0) = \theta_0$ and $\dot{\theta}(t = 0) = \dot{\theta}_0$, plot both $\theta(t)$ and $\dot{\theta}(t)$ as a function of time. Plot the orbit $\{\theta, \dot{\theta}\}$ from your numerical solution and compare it with the one obtained from the energy conservation.

Create a different figures or sub-plots for each plot and label the axis correctly.