Exercises week 3

October 19, 2017

Please submit your work before the following class (name the script file as `your_name_problems-week3.m`) at the following e-mail address `francesca.marchetti@uam.es`.

1 Position, velocity and acceleration

Let's suppose that of the motion of a particle in a one-dimensional line, we know its velocity as a function of time

$$v(t) = v_0 + V_a \cos(\omega t) e^{-t\gamma} , \quad (1)$$

where the parameters are: $v_0 = 2.3$ m/s, $V_a = 10$ m/s, $\omega = 2$ s^{-1}, and $\gamma = 0.2$ s^{-1} — careful about the units.

1. Plot the particle velocity $v(t)$ in the interval $t \in [t_{min}, t_{max}]$, where $t_{min} = 0$ s and $t_{max} = 10$ s N.B. label the plot axis with the correct units.
2. Numerically evaluate the particle acceleration $a(t)$ for the same interval of time and plot it (axis labels and units); compare the numerical result you get with the analytical one.
3. Numerically evaluate the particle position $x(t)$ in the same interval of time, by integrating (1) and knowing that $x(t_{min}) = x0 = 1$ m. Plot the position $x(t)$ for $t \in [t_{min}, t_{max}]$.

Hints to solve Exercise 1

- Position $x(t)$, velocity $v(t)$ and acceleration $a(t)$ of a particle that moves on a line (one dimension) are related by

$$v(t) = \frac{dx(t)}{dt} \qquad a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} . \quad (2)$$

- Thus, if the velocity is known in a certain interval of time and one wants to evaluate the position at a given time in that interval, one has to evaluate the following integral:

$$x(t) = x(t_0) + \int_{t_0}^{t} ds v(s) , \quad (3)$$

where $x(t_0)$ is the position of the particle at the time $t = t_0$.

2 Exercise:

The equation of motion for the one-dimensional harmonic oscillator can be derived from the Newton's law $F = ma = m\frac{d^2x}{dt^2}$ and the Hooke's law, $F = -kx$, describing the motion of an object of mass m which, displaced from its equilibrium position by a distance x, experiences a restoring force proportional to the displacement:

$$m\frac{d^2x(t)}{dt^2} + kx(t) = 0 \; . \tag{4}$$

This differential equation, with initial conditions $x(t = 0) = x_0$ and $v(t = 0) = v_0$, where $v(t) = \frac{dx(t)}{dt}$ is the velocity, admits the exact solution

$$x(t) = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}} \cos(\omega_0 t + \phi) \; , \tag{5}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ and $\phi = \arccos \frac{x_0}{\sqrt{x_0^2 + v_0^2/\omega_0^2}}$.

1. Plot the position $x(t)$ in the interval of time $t \in [0, 4\pi/\omega_0]$ (label the plot axis with the correct units) with the initial conditions $x_0 = 3.2$ m and $v_0 = -2$ m/s, and for the following values of the system parameters: mass $m = 2$ kg and spring constant $k = 4$ kg/s^2;

2. evaluate numerically the velocity $v(t)$ and plot the numerical result together with the analytical one;

3. evaluate numerically the acceleration $a(t)$ and plot the numerical result together with the analytical one;

4. evaluate and plot as a function of time the kinetic energy $T = \frac{1}{2}mv^2$, the potential $U = \frac{1}{2}kx^2$, and the total energy $E = T + U$. Comment the result you get.

3 Exercise:

A mass m, connected to a spring and submerged in a fluid, obeys the Newton equation for a damped harmonic oscillator:

$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} + kx(t) = 0 \ , \tag{6}$$

where c is the friction constant and k the elastic spring constant. For initial conditions of motion $x(t = 0) = x_0$ and $v(t = 0) = v_0 = 0$, where $v(t) = \frac{dx(t)}{dt}$ is the velocity, the exact solution of this equations reads as

$$x(t) = \frac{x_0}{\sqrt{1-\zeta^2}} e^{-\gamma t} \cos(\omega_0 t\sqrt{1-\zeta^2} - \varphi) \ , \tag{7}$$

where

$$\gamma = \frac{c}{2m} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$
$$\zeta = \frac{c}{2\sqrt{mk}} \qquad \varphi = \arccos(\sqrt{1-\zeta^2}) \ .$$

1. Create an external function `[position]=dho_position(m, c, k, x0, t)` for the mass position and plot the solution in the interval $t \in [t_0, t_1] = [0, 18]$ s in the underdamped case where $m = 1.4$ kg, $k = 6.5$ N/m, $c = 0.8$ kg/s and with initial condition $x_0 = 2.8$ m;

2. By building an external function routine for derivation, evaluate and plot the velocity $v(t)$ and the acceleration $a(t)$;

3. Check numerically that the equation of motion (6) is satisfied by this solution (7);

4. By building an external function routine for integration, evaluate and plot the work done by the friction against the motion, $W = \int dx(-cv) = -c\int_{t_0}^{t_1} dt v^2$;

5. Evaluate and plot the kinetic energy $T = \frac{1}{2}mv^2$, the potential energy $U = \frac{1}{2}kx^2$ and the total energy $E = T + U$;

6. Consider the overdamped case $m = 1$ kg, $k = 1$ N/m, $c = 4$ kg/s and with initial condition $x_0 = 0$ m and repeat the above analysis.

4 Exercise:

Write a script that evaluates the factorial `n!` of a given natural number `n` and compare the results with the built-in function `factorial(n)` — remember that the factorial is defined as $n! = n(n-1)(n-2)\ldots 2 * 1$; Hint: store the result in a variable `f` that needs to be initialised to `f=1` prior to the loop.

5 Exercise:

Use the loop `while` to generate a routine which is able to establish whether a natural number n is a prime number (i.e., a natural number which can only be divided by 1 and itself) or not. Rewrite the same routine with the loop `for`.

Hints to solve Exercise 5

- You can use the command `rem(n,div)` (or equivalently the command `mod(n,div)`) to evaluate the rest of a division; if an integer number `div` exists such that `mod(n,div)=0`, then clearly it means that `n` cannot be a prime number;
- remember the use of `if` (see previous section);
- you can terminate a loop like `while` and `for` with the command `break` (e.g., once the conditions you are looking for are met, without the need to leave the computer run till the natural end the loop; in this way you shorten the evaluation time);
- once you have established the conditions you are looking for, you can print "n is a prime number" (or "n is not a prime number") with the command `disp('n is a prime number')`.

6 Rotation and shifting of parametric curves

Let's consider the following ellipse

$$x = a\sin(\theta) \qquad\qquad y = b\cos(\theta)\ ;$$

where $\theta \in [0, 2\pi]$ and $a = 2$ and $b = 1$.

1. Plot the ellipse (use the command `axis('equal')` to have the same axis ranges for both x and y).
2. Plot now the ellipse rotated clockwise by an angle $\theta = \pi/8$ and shifted by $(x_0, y_0) = (2, 3)$, i.e., now centered in $(x_0, y_0) = (2, 3)$.

7 Numerical series and asymptotic limits

The number e can be equivalently defined as

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \ .$$

1. Find an estimate of e by using both the definitions given above and compare them with the built-in value of Matlab (or `exp(1)`) — for summing the vector components you can use the command `sum` or you can find an equivalent way of doing it by using the multiplication operation of appropriate vectors.

2. Consider the finite sum

$$e(N) = \sum_{n=0}^{N} \frac{1}{n!} \ ,$$

   and plot it in two subplots both as a function of N and N^{-1}, showing that it asymptotically converges to e.

8 Exercise:

Let us consider the geometric series

$$a_{exact} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for } |x| < 1 \ . \tag{8}$$

1. Choose a numerical value for x (say $x = 1/2$) and plot the sum of the first N terms of the geometric series, i.e., it's approximated numerical value

$$a(N) = \sum_{n=0}^{N} x^n \ , \tag{9}$$

   as a function of N and check that it converges to $1/(1-x)$. Compare the values you get for a fixed N with the exact analytical expression $a(N) = \frac{1-x^{N+1}}{1-x}$.

2. Plot now $a(N)$ as a function of N^{-1}.

9 Exercise:

Consider the following numerical series (Leibniz formula) defining π:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \ . \tag{10}$$

1. Evaluate the finite sum

$$f(N) = \sum_{n=0}^{N} \frac{(-1)^n}{2n+1} \ , \tag{11}$$

plot it as a function of N and check that it converges to $\pi/4$;

2. Consider now the equivalent form

$$g(N) = \sum_{n=0}^{N} \frac{2}{(4n+1)(4n+3)} \ , \tag{12}$$

plot it and check it converges to the same value $\pi/4$.