Exercises week 5

November 16, 2017

Please submit your work before the following class (name the script file as `your_name_problems-week5.m`) at the following e-mail address `francesca.marchetti@uam.es`.

1 Exercise:

Solve numerically the following first order differential equation

$$\frac{dx(t)}{dt} = -(3t^2 - 2t + 5)[x(t) - 1] \qquad\qquad x(0) = x0 \ ,$$

for three different initial conditions, $x(0) = 0$, $x(0) = 1$, and $x(0) = 2$ and compare the numerical results with the exact solution.

2 Exercise:

Consider the second order differential equation describing an harmonic oscillator:

$$m\frac{d^2x}{dt^2} = -\kappa x \ .$$

This equation can be rewritten in terms of the dimensionless time $\tilde{t} = \omega_0 t$, where $\omega_0 = \sqrt{\kappa/m}$, as

$$\frac{d^2x}{d\tilde{t}^2} = -x \ . \tag{1}$$

Solve this equation with the following initial conditions

$$x(0) = 1 \qquad\qquad v(0) = \left.\frac{dx}{d\tilde{t}}\right|_{\tilde{t}=0} = 0 \ ,$$

applying the Euler method.

Questions:

1. How the numerical result compares with the exact one in the interval $\tilde{t} \in [0, 4\pi]$?

2. roughly how many points N do you need to consider in the interval $\tilde{t} \in [0, 4\pi]$ so that to have a good numerical approximation to the exact solution?

3. in a separate figure, plot the velocity $v(\tilde{t})$ as a function of the position $x(\tilde{t})$ (phase space). Why the exact solution gives a closed trajectory? Why the numerical solution is not a closed trajectory?

Hints to solve Exercise 2

- The exact solution of the harmonic oscillator in dimensionless units is given by

$$x(t) = \sqrt{x^2(0) + v^2(0)}\sin(\tilde{t} + \delta) \qquad \delta = \arctan\left(\frac{x(0)}{v(0)}\right) \ . \tag{2}$$

  Note that if $v(0) = 0$, then $\delta = \pi/2$;

- the 2^{nd}–order ordinary differential equation (1) can be written as a system of two 1^{st}–order ordinary differential equations:

$$\begin{cases} \dfrac{dx}{d\tilde{t}} = v \\ \dfrac{dv}{d\tilde{t}} = -x \ . \end{cases}$$

  In this case the Euler method reads

$$\begin{cases} x_{i+1} = x_i + \delta t v_i \\ v_{i+1} = v_i + \delta t(-x_i) \ . \end{cases}$$

3 Exercise:

Solve the following differential equation

$$\frac{dx}{dt} = 5x - x^2 \qquad\qquad x(t=0) = 1 \ , \tag{3}$$

with the three approximated methods we have so far introduced (Euler, modified Euler, and second order Runge-Kutta) and compare the approximated solutions you get with the exact one:

$$x(t) = \frac{5e^{5t}}{4 + e^{5t}} \ . \tag{4}$$