*Few Problems on "**Superfluidity in Ultracold Fermi Gases**"*
(1 tutorial class at "Physics by the Lake")

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All the problems (with some of their answers) are contained in the lecture notes (boxes). Here, you can find few selected problems for the one tutorial class on *Superfluidity in Ultracold Fermi Gases.* Answers for these problems can be found separately.

1 The ideal Fermi gas

Making use of the local density approximation (LDA),

$$\mu(\mathbf{r}) = V(\mathbf{r}) - \mu\ . \tag{1}$$

evaluate the expression of the Fermi energy and temperature for a trapped gas:

$$k_B T_F = \varepsilon_F = \bar{\omega}(6N)^{1/3} \simeq 1.82\bar{\omega}N^{1/3}\ . \tag{2}$$

2 BCS theory

Show that the mean-field Hamiltonian

$$\hat{H} - \varepsilon_F \hat{N} \simeq \sum_{\mathbf{k}} \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} & c_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{k}} & -\Delta \\ -\Delta & -\xi_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} + \sum_{\mathbf{k}} \xi_{\mathbf{k}} - \frac{\Delta^2}{g} V\ . \tag{3}$$

can be diagonalised by making use of the unitary transformation:

$$\begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix}\ . \tag{4}$$

Show that the unitary transformation conserves the anti-commutation relations for the (Bogoliubov) quasi-particle operators $\gamma_{\mathbf{k}\sigma}$ and that the quasi-particle energy is given by $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$, i.e.:

$$\hat{H} - \varepsilon_F \hat{N} \simeq \sum_{\mathbf{k},\sigma=\uparrow,\downarrow} E_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}}) - \frac{\Delta^2}{g} V\ . \tag{5}$$

3 BEC-BCS crossover

(a) Show that, as originally proposed by Leggett, the ground state

$$|\psi\rangle = \prod_{\mathbf{k}} \left(\cos\theta_{\mathbf{k}} + \sin\theta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right) |0\rangle\ , \tag{6}$$

where $\theta_{\mathbf{k}} = \theta_{-\mathbf{k}}$, interpolates between a BEC condensate of tightly bound molecules (when $1/k_F a \to +\infty$) to a BCS state (when $1/k_F a \to -\infty$).

(b) For a contact potential, show that in the BCS limit $1/k_F a \to -\infty$ ($\Delta \ll \varepsilon_F \simeq \mu$) the gap equation can be solved to give

$$\Delta = \frac{8}{e^2} \varepsilon_F e^{-\pi/2|a|k_F}\ , \tag{7}$$

while in the BEC limit $1/k_F a \to +\infty$ ($\varepsilon_F \ll \Delta \ll |\mu|$) one reobtains from

$$\frac{m}{4\pi a} = \frac{1}{V} \sum_{\mathbf{k}}^{k_0} \left(\frac{1}{2\epsilon_{\mathbf{k}}} - \frac{1}{2E_{\mathbf{k}}} \right) , \tag{8}$$

the energy of the molecular bound state:

$$\mu = \frac{\epsilon_b}{2} = -\frac{1}{2ma^2} . \tag{9}$$

4 Magnetised superconductor

Show that for a magnetised superconductor (with attractive coupling constant $g \equiv -\lambda < 0$), the gap equation reads

$$\Delta = \lambda \mathcal{N}(\varepsilon_F) \int_{\sqrt{\max\{0, h^2 - \Delta^2\}}}^{\omega_D} \frac{d\xi}{2\sqrt{\xi^2 + \Delta^2}} , \tag{10}$$

(i.e., if $h > \Delta$ the quasi-particle states are occupied only for $\xi > \sqrt{h^2 - \Delta^2}$) and that admits the following solution:

$$\Delta = \Delta_{\text{BCS}} \begin{cases} 1 & h < \Delta \\ \sqrt{2h - \Delta_{\text{BCS}}} \equiv \Delta_{\text{Sarma}} & h > \Delta . \end{cases} \tag{11}$$

where Δ_{BCS} is the BCS solution (1.31) (see Fig. 3.4).