Unit 5

November 29, 2015

1 Ordinary differential equations

An ordinary differential equation of order n is an equation involving a single independent variable x, a function $f(x)$ and the derivative of such a function up to the n^{th}–derivative,

$$g\left(\frac{d^n f(x)}{dx^n}, \frac{d^{n-1} f(x)}{dx^{n-1}}, \dots \frac{df(x)}{dx}, f(x), x\right) = 0 \ , \tag{1}$$

where g is some arbitrary function. The order of the equation is set by the highest derivative, $\frac{d^n f(x)}{dx^n}$. In addition, the differential equation (1) is called ordinary because there is only one independent variable, x. The form of (1) is implicit, but in some cases it can be rewritten in an explicit form:

$$\frac{d^n f(x)}{dx^n} = h\left(\frac{d^{n-1} f(x)}{dx^{n-1}}, \dots \frac{df(x)}{dx}, f(x), x\right) \ . \tag{2}$$

The example we are going to use the most through this unit is the one of the Newton's second law of motion for a particle of mass m moving under the influence of a force F. The force in general can be a function of the position $x(t)$ of the particle, but it can also be a function of it's velocity $v(t) = dx(t)/dt$ (like in the case when a particles suffers friction) and time t (like for other non-conservative forces, such as time–dependent driving forces):

$$m\frac{d^2 x(t)}{dt^2} = F\left(x(t), \frac{dx(t)}{dt}, t\right) \ . \tag{3}$$

This is a second order explicit ordinary differential equation. Most of equations in physics are differential equations telling us how certain quantities change as functions of others; in the case of the Newton equation (3), it tells us how the position $x(t)$ of a particle of mass m changes with time t.

1.1 Reduction of order

Any ordinary differential equation of order n can be rewritten as a system of n ordinary differential equations of order 1. For example, in the case of Newton's second law of motion (3), we can introduce the velocity field $v(t)$,

$$\begin{cases} v(t) = \dfrac{dx(t)}{dt} \\ m\dfrac{dv(t)}{dt} = F(x(t), v(t), t) \ , \end{cases} \tag{4}$$

and thus, rather than a second order explicit ordinary differential equation for the position $x(t)$ (3), we have now two explicit first order differential equations for both the position $x(t)$ and the velocity $v(t)$. Therefore we will first introduce numerical techniques for solving 1^{st}–order ordinary differential equations and later on we will see how to apply such techniques in order to solve the system (4).

2 First order ordinary differential equations: Euler method

The simplest method for numerical integration of a first order ordinary differential equation is given by the Euler method. As we will see later, as well as very simple, this method is also not very 'accurate' and we will consider better methods later on. Let us suppose we want to solve the following 1^{st}–order ordinary differential equation, with a given initial condition for $t = t_1$:

$$\frac{dx(t)}{dt} = f\left(x(t), t\right) \qquad\qquad x(t_1) = x_1 \; . \tag{5}$$

Basically we know the value of the function at a given point (i.e., $x_1 = x(t_1)$) and the slope of the tangent to $x(t)$ at the same point (i.e., $\left.\frac{dx(t)}{dt}\right|_{t=t_1} = f(x_1, t_1)$) and from this information we want to reconstruct the entire solution $x(t)$. Starting from the initial condition, $x(t_1) = x_1$, we can evaluate the solution at a point t_2 close to t_1 by approximating the derivative to the first order,

$$\frac{x_2 - x_1}{t_2 - t_1} \simeq f(x_1, t_1) \qquad \Rightarrow \qquad x_2 \simeq x_1 + \delta t f(x_1, t_1) \; ,$$

where $x_2 = x(t_2)$ and $\delta t = t_2 - t_1$. The smaller δt, the more accurate the approximation we are doing is. How to find x_2 is also schematically plotted in Fig. 1.

Generalising this expression to the case in which we consider a grid of N points in the interval $[t_1, t_N]$ in which we want to evaluate the solution to the first order differential equation (5), we can write that (see Fig. 1):

$$\frac{x_{i+1} - x_i}{t_{i+1} - t_i} \simeq f(x_i, t_i) \qquad \Rightarrow \qquad x_{i+1} \simeq x_i + \delta t f(x_i, t_i) \; , \tag{6}$$

where $x_i = x(t_i)$, $x_{i+1} = x(t_{i+1})$, and $\delta t = t_{i+1} - t_i$. A schematic representation of the Euler method is shown in Fig. 1. The structure of the Euler algorithm can be written as follows

```
t1= ...; tN= ...; N=...;
t=linspace(t1, tN, N);
dt=t(2)-t(1);
% initial condition:
x_E(1)=...
% anonymous function f(x,t):
f=@(x,t)(...);
% loop for the Euler method:
for i=1:N-1
            x_E(i+1)=x_E(i) + dt*f(x_E(i), t(i));
end
```

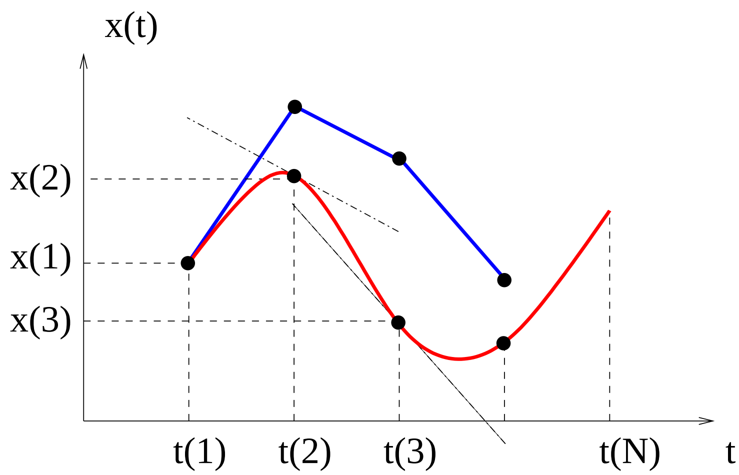

Figure 1: Schematic representation of the Euler method. The exact solution to a given differential equation satisfying the initial condition $x(t_1) = x_1$ is plotted in red and in blue is instead plotted the approximated solution obtained by using the Euler method.

Alternatively you can evaluate the values of the function `f(i)=...` inside the loop. Once you have become familiar with the method, you are invited to write an external script function routine for the Euler method, as in the following example:

```
%-----------------------------------------------%
% Euler_method                                  %
%-----------------------------------------------%
% INPUTs (to be provided):
% t1 = starting point
% x1 = initial condition
% tN = end point
% N = number of steps
% xp = anonymous function containing the first derivative
% OUTPUTs:
% xt_E = vector solution x(t)
% t = grid of point t where x(t) has been evaluated

function [xt_E, t] = Euler_method(t1, x1, tN, N, xp)
xt_E(1)=x1;
t=linspace(t1,tN,N);
dt=t(2)-t(1);
for i=1:N-1
    xt_E(i+1)=xt_E(i) + dt*xp(xt_E(i),t(i));
end
end
```

2.0.1 Exercise

Consider the following first order differential equation

$$\frac{dx}{dt} = -2\left(t - \frac{5}{2}\right) ,$$

with the initial condition $x(1) = -9/4$. Solve the equation exactly and compare the exact solution with the approximated one obtained with the Euler method. The approximated solution evaluated on a grid of $N = 5$ points within the interval $[1, 4]$ is shown in Fig. 2.

Questions:

1. if $x(t)$ is the position of a particle (of mass $m = 1$kg) in meters, t the time in seconds, evaluate the acceleration of the particle and describe its motion — N.B. a freely falling body on the moon has an acceleration of 1.6m/s^2.

After you have completed the exercise, solve it again by writing an external function routine for the part containing the numerical integration via the Euler method.

Hints to solve Exercise 2.0.1

- In order to find the exact solution you have to evaluate

$$\int_{t_1=1}^{t} dt' \frac{dx(t')}{dt'} = x(t) - x(t_1) = \int_1^t dt' \left[-2\left(t' - \frac{5}{2}\right)\right] = -\left(t - \frac{5}{2}\right)^2 + \left(1 - \frac{5}{2}\right)^2 .$$

  Thus the exact solution is $x(t) = -\left(t - \frac{5}{2}\right)^2$.

2.0.2 Exercise

Consider the following first order differential equation

$$\frac{dx}{dt} = -3x , \tag{7}$$

with the initial condition $x(0) = 1$. Solve the equation exactly and compare the exact solution with the approximated one obtained with the Euler method. Describe the motion in time t of a particle with position $x(t)$ — N.B. a force proportional to the velocity of a particle and with opposite direction,

$$F = m\frac{d^2x}{dt^2} = -|\alpha|\frac{dx}{dt} ,$$

describes the friction on the particle. The solution you get should look like the one plotted in Fig. 3.
After you have completed the exercise, solve it again by using the very same external function routine for the Euler method that you have written for the previous exercise 2.0.1.

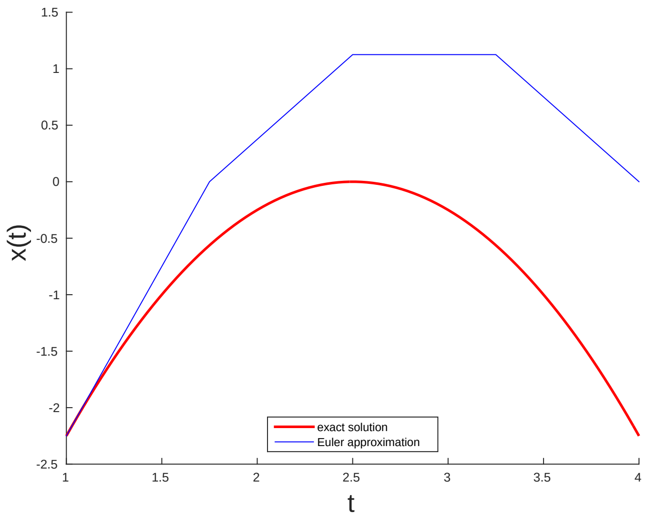

Figure 2: Solutions of the of the differential equation $dx/dt = -2(t - 5/2)$ with initial condition $x(1) = -9/4$: exact solution (red) and approximated solution (blue) obtained with the Euler method on a grid of $N = 5$ points in the interval $[1, 4]$.

Hints to solve Exercise 2.0.2

- An exact solution of (7) can be found by evaluating

$$\int_{x(0)}^{x} \frac{dx'}{x'} = \ln\left(\frac{x(t)}{x(0)}\right) = -3\int_0^t dt' = -3t\ .$$

2.0.3 Exercise

Solve numerically the following first order differential equation

$$\frac{dx(t)}{dt} = -(3t^2 - 2t + 5)[x(t) - 1] \qquad\qquad x(0) = x0\ ,$$

for three different initial conditions, $x(0) = 0$, $x(0) = 1$, and $x(0) = 2$ and compare the numerical results with the exact solution.

3 Second order ordinary differential equations: Euler method

Let us consider the particular case of the differential equation for an harmonic oscillator, like the one describing the motion of a particle of mass m, which, displaced from its equilibrium position, experiences a restoring force proportional to the displacement x, $F = -\kappa x$ (Hooke's law):

$$m\frac{d^2x}{dt^2} + \kappa x = 0\ , \tag{8}$$

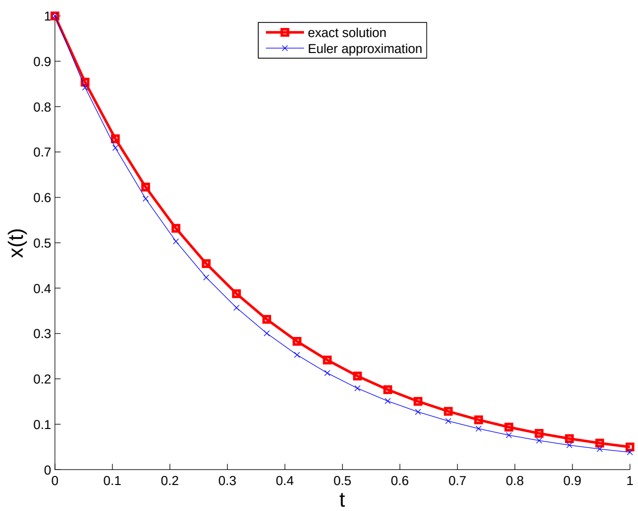

Figure 3: Solutions of the of the differential equation $dx/dt = -3x$ with initial condition $x(0) = 1$: exact solution (red squares) and approximated solution (blue crosses) obtained with the Euler method on a grid of $N = 20$ points in the interval $[0, 1]$.

As already explained previously in Sec. 1.1, we can rewrite this second order ordinary differential equation as a system of two coupled first order differential equations:

$$\begin{cases} \dfrac{dx}{dt} = g_1(x, v, t) = v \\ \dfrac{dv}{dt} = g_2(x, v, t) = -\dfrac{\kappa}{m} x \; . \end{cases} \tag{9}$$

In this way, we can solve the initial problem by making use of the Euler method, as in the following exercise.

3.0.4 Exercise: the harmonic oscillator

Consider the second order differential equation describing an harmonic oscillator:

$$m\frac{d^2x}{dt^2} = -\kappa x \; .$$

This equation can be rewritten in terms of the dimensionless time $\tilde{t} = \omega_0 t$, where $\omega_0 = \sqrt{\kappa/m}$, as

$$\frac{d^2x}{d\tilde{t}^2} = -x \; . \qquad (10)$$

Solve this equation with the following initial conditions

$$x(0) = 1 \qquad\qquad v(0) = \left.\frac{dx}{d\tilde{t}}\right|_{\tilde{t}=0} = 0 \; ,$$

applying the Euler method.

Questions:

1. How the numerical result compares with the exact one in the interval $\tilde{t} \in [0, 4\pi]$?
2. roughly how many points N do you need to consider in the interval $\tilde{t} \in [0, 4\pi]$ so that to have a good numerical approximation to the exact solution?
3. in a separate figure, plot the velocity $v(\tilde{t})$ as a function of the position $x(\tilde{t})$ (phase space). Why the exact solution gives a closed trajectory? Why the numerical solution is not a closed trajectory?

Hints to solve Exercise 3.0.4

- The exact solution of the harmonic oscillator in dimensionless units is given by

$$x(t) = \sqrt{x^2(0) + v^2(0)}\sin(\tilde{t} + \delta) \qquad \delta = \arctan\left(\frac{x(0)}{v(0)}\right) \; . \qquad (11)$$

  Note that if $v(0) = 0$, then $\delta = \pi/2$;

- the 2^{nd}–order ordinary differential equation (10) can be written as a system of two 1^{st}–order ordinary differential equations:

$$\begin{cases} \dfrac{dx}{d\tilde{t}} = v \\ \dfrac{dv}{d\tilde{t}} = -x \; . \end{cases}$$

  In this case the Euler method reads

$$\begin{cases} x_{i+1} = x_i + \delta t v_i \\ v_{i+1} = v_i + \delta t(-x_i) \; . \end{cases}$$

3.1 Error in the Euler method

At each step in the Euler method (6) we are neglecting terms of the order of δt^2, therefore we say the 'local' error is of the order $(\delta t)^2$. However, each step is iterated $N-1$ times, where N is the number of points on the grid. Therefore the magnitude of the 'global' Euler error is given by

$$\text{error} \sim (\delta t)^2 \times (N-1) \; . \tag{12}$$

As N is of the order of $1/\delta t$ then the Euler method global error is of the order of δt. For this reason, the Euler method is a **first order** method. As the following exercise shows, the convergence to the exact solution is quite slow.

3.1.1 Exercise about the error in the Euler method

Evaluate the error done with the Euler method in the previous three exercises by using the estimate (12) and compare this error with the one you obtained evaluating the maximum distance of the approximated solution from the exact one. Choosing one of the three previous exercises, evaluate the behaviour of the error (12) with increasing the number of points N on the grid. How fast is the convergence to the exact solution reached with the Euler method?

4 First order differential equations: modified Euler method

In the previous classes, we have seen how to solve a 1^{st}–order differential equation with an assigned initial condition,

$$\frac{dx(t)}{dt} = f\left(x(t), t\right) \qquad\qquad x(t_1) = x_1 \; . \tag{13}$$

via the Euler method. In particular, if we consider a grid of N points in the interval $[t_1, t_N]$ where we want to evaluate the solution, the solution at the time t_{i+1}, $x_{i+1} = x(t_{i+1})$, can be evaluated approximatively (to the first order) starting from the solution at the time t_i, $x_i = x(t_i)$ and adding to it the derivative of such a solution in t_i, $f(x_i, t_i)$, times the increment in time $\delta t = t_{i+1} - t_i$:

$$x_{i+1} \simeq x_i + \delta t f(x_i, t_i) \; . \tag{14}$$

We have seen that the Euler method is a first order method, which means the error made is of the order of δt — note that $\delta t \sim (\delta t)^2 \times (N-1)$, where N is the number of times the method is iterated (the error accumulates at each iteration step). For this reason this method converges slowly, and is, in some cases, unstable.

A better accuracy could be obtained by considering a higher order approximation to the derivative, i.e., instead of using the slope $f(x_i, t_i)$, we use the average slope $[f(x_i, t_i) + f(x_{i+1}, t_{i+1})]/2$:

$$x_{i+1} \simeq x_i + \delta t \frac{f(x_i, t_i) + f(x_{i+1}, t_{i+1})}{2} \; . \tag{15}$$

However, the problem is that on the right-hand side of this formula it appears x_{i+1} that we don't know yet, because at each step we only know x_i and t_i. To solve this problem, the modified Euler method first calculates the intermediate value $\tilde{x}_{i+1}$ by means of the Euler method, and then uses this value for the final approximation x_{i+1} at the next integration point, in other words:

$$\tilde{x}_{i+1} = x_i + \delta t f(x_i, t_i) \tag{16}$$

$$x_{i+1} = x_i + \delta t \frac{f(x_i, t_i) + f(\tilde{x}_{i+1}, t_{i+1})}{2} \ . \tag{17}$$

5 First order differential equations: Runge-Kutta method

Another second order method is given by the 2^{nd}–order Runge–Kutta method. The idea this method is based on is similar to the one of the modified Euler method explained before. In fact, an equivalent way of using a better approximation than (14) consists in replacing the derivative evaluated at the left–point of the interval $[t_i, t_{i+1}]$, $f(x_i, t_i)$, with the derivative evaluated at the middle–point, $f(x_{i+1/2}, t_{i+1/2})$:

$$x_{i+1} \simeq x_i + \delta t f(x_{i+1/2}, t_{i+1/2}) \ . \tag{18}$$

The notation in this expression means $t_{i+1/2} \equiv t_i + \delta t/2$ and $x_{i+1/2} \equiv x(t_{i+1/2})$. The problem with the expression (18) is that the algorithm cannot be applied in the form (18) since it requires the knowledge of the derivative evaluated at the middle–point, $f(x_{i+1/2}, t_{i+1/2})$, which we don't know because at each step we only know x_i and t_i. We can however approximate $x_{i+1/2}$ by again using (14), i.e., the Euler method:

$$x_{i+1/2} \simeq x_i + \frac{\delta t}{2} f(x_i, t_i) \ .$$

Therefore the 2^{nd}–order Runge–Kutta algorithm reads as:

$$x_{i+1} \simeq x_i + \delta t f(x_i + \frac{\delta t}{2} f(x_i, t_i), t_i + \delta t/2) \ . \tag{19}$$

We will see later in Sec. 3.1 why this method is second order rather than first.

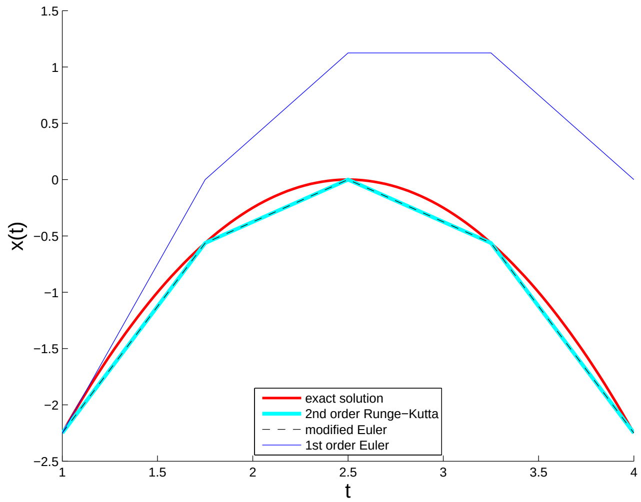

Figure 4: Solutions of the of the differential equation $dx/dt = -2(t - 5/2)$ with initial condition $x(t = 1) = -9/4$. Exact solution (red) and approximated solutions: in blue the one obtained with the 1^{st}–order Euler method, in cyan the one obtained with a 2^{nd}–order Runge-Kutta method, and in black dashed line the one obtained with the modified Euler method (it gives exactly the same result as the 2^{nd}–order Runge-Kutta method) on a grid of $N = 5$ points in the interval $[1, 4]$.

6 Exercises

6.0.2 Exercise

Consider the following first order differential equation

$$\frac{dx}{dt} = -2\left(t - \frac{5}{2}\right) ,$$

with the initial condition $x(t = 1) = -9/4$. Evaluate the approximated solutions obtained with the Euler method, the modified Euler method and with the 2^{nd}–order Runge–Kutta method and compare them with the exact solution $x(t) = -(t - 5/2)^2$. One example is shown in Fig. 4 for a grid of $N = 5$ points within the interval $[1, 4]$.

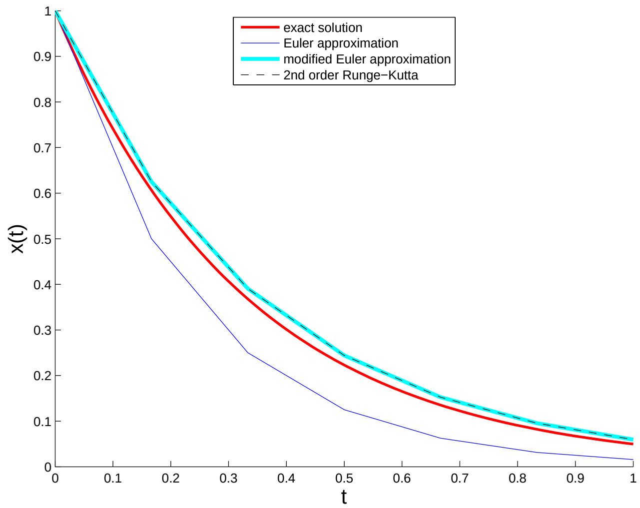

Figure 5: Solutions of the of the differential equation $dx/dt = -3x$ with initial condition $x(t = 0) = 1$. Exact solution (red) and approximated solutions: in blue the one obtained with the 1^{st}–order Euler method, in cyan the one obtained with a 2^{nd}–order Runge-Kutta method, and in black dashed line the one obtained with the modified Euler method on a grid of $N = 7$ points in the interval $[0, 1]$.

6.0.3 Exercise

Consider the following first order differential equation

$$\frac{dx}{dt} = -3x \,, \tag{20}$$

with the initial condition $x(t = 0) = 1$. Compare the exact solution to this equation with the approximated ones obtained with the Euler method, the modified Euler method and the second order Runge-Kutta method — see Fig. 5.
After you have completed this and the previous exercise, solve them again by building two external function routines for the modified Euler method and the second order Runge-Kutta method, respectively.

6.0.4 Exercise:

Solve the following differential equation

$$\frac{dx}{dt} = 5x - x^2 \qquad\qquad x(t=0) = 1 , \tag{21}$$

with the three approximated methods we have so far introduced (Euler, modified Euler, and second order Runge-Kutta) and compare the approximated solutions you get with the exact one:

$$x(t) = \frac{5e^{5t}}{4 + e^{5t}} . \tag{22}$$

6.0.5 Exercise

Solve numerically the following first order differential equation with the second order Runge-Kutta method

$$\frac{dx(t)}{dt} = -(3t^2 - 2t + 5)[x(t) - 1] \qquad\qquad x(0) = x0 ,$$

for three different initial conditions, $x(0) = 0$, $x(0) = 1$, and $x(0) = 2$ and compare the numerical results you obtain with the exact solution.

6.1 Error in the Runge–Kutta method

It easy to understand why, by taking the middle point, the approximation we do is now second order rather than first. Let us consider for simplicity the case in which $f(x,t) = f(t)$ therefore the solution of the differential equation can now be obtained by direct integration,

$$x(t) - x(t_0) = \int_{t_0}^{t} dt' f(t') ,$$

and the solution at the point t_{n+1} can be exactly derived by the one at t_n by

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} dt' f(t') . \tag{23}$$

If, in the interval $[t_n, t_{n+1}]$ we approximate the function $f(t')$ by it's middle point $f(t') \simeq f(t_{n+1/2}) + (t' - t_{n+1/2})f'(t_{n+1/2}) + O(\delta t^2)$, we can notice that substituting back into the expression (23) the term

$$\int_{t_n}^{t_{n+1}} dt'(t' - t_{n+1/2}) = 0 ,$$

is zero. Therefore automatically by considering the middle point, we are doing an error in the solution x_{n+1} which is of the order of δt^2.

7 Application: Harmonic oscillator, friction, and external drive

7.0.1 Exercise: the harmonic oscillator

Consider the equation of motion for an undamped spring–mass,

$$\frac{d^2x}{d\tilde{t}^2} = -x \; , \tag{24}$$

and solve this equation with the following initial conditions

$$x(0) = 1 \qquad\qquad v(0) = \left.\frac{dx}{d\tilde{t}}\right|_{\tilde{t}=0} = 0 \; ,$$

applying the Runge–Kutta method.

Questions:

1. Compare the Runge–Kutta numerical solution with the solution obtained with the Euler algorithm and with the exact one, $x(t) = \cos(\tilde{t})$;
2. for a given number of points N in the interval $\tilde{t} \in [0, 4\pi]$ which approximations is closer to the exact one?
3. evaluate the errors for the Euler method, `max(abs(xRK-xexact))`, and the one for the Runge–Kutta method, `max(abs(xE-xexact))` with increasing the number of points N and observes which method gives the fastest convergence — you need to plot the error versus N in a log-log scale to see which curve has the steepest slope.

Hints to solve Exercise 7.0.1

- You can write a 2^{nd}–order differential equation,

$$\frac{d^2x}{dt^2} = g(x, \frac{dx}{dt}) \; ,$$

  as a system of two 1^{st}–order differential equations:

$$\begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt} = g(x, v) \; ; \end{cases}$$

- for such equations the 2^{nd}–order Runge–Kutta method writes

$$\begin{cases} x_{i+1} \simeq x_i + \delta t v_{i+1/2} \\ v_{i+1} \simeq v_i + \delta t g(x_{i+1/2}, v_{i+1/2}) \; ; \end{cases}$$

- the middle–point values, $v_{i+1/2}$ and $x_{i+1/2}$ can be approximated as:

$$\begin{cases} v_{i+1/2} \simeq v_i + \dfrac{\delta t}{2} g(x_i, v_i) \\ x_{i+1/2} \simeq x_i + \dfrac{\delta t}{2} v_i \ . \end{cases}$$

The damped harmonic oscillator, such as a mass m, connected to a spring and submerged in a fluid, experiences a frictional force, which can be modeled as a force proportional, and opposite in direction, to the oscillator velocity:

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0 \frac{dx(t)}{dt} + \omega_0^2 x(t) = 0 \qquad v(0) = v_0 \qquad x(0) = x_0 \ , \tag{25}$$

where $\omega_0 = \sqrt{\dfrac{k}{m}}$ and $\zeta = \dfrac{c}{2\sqrt{mk}}$.

7.0.2 Exercise

Solve numerically the damped harmonic oscillator (25) in the interval $t \in [t_0, t_1] = [0, 18]$ s for the underdamped case where $m = 1.4$ kg, $k = 6.5$ N/m, $c = 0.8$ kg/s and with initial conditions $x_0 = 2.8$ m and $v_0 = 0$ and compare the numerical result with the exact one

$$x(t) = \frac{x_0}{\sqrt{1-\zeta^2}} e^{-\gamma t} \cos(\omega_0 \sqrt{1-\zeta^2} - \varphi) \ ,$$

where $\gamma = \dfrac{c}{2m}$ and $\varphi = \arccos(\sqrt{1-\zeta^2})$.

In presence of an external drive $F(t)$, the equation of motion of a damped harmonic oscillator reads as:

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0 \frac{dx(t)}{dt} + \omega_0^2 x(t) = \frac{F(t)}{m} \qquad v(0) = v_0 \qquad x(0) = x_0 \ . \tag{26}$$

7.0.3 Exercise

Consider the case of an external sinusoidal drive, $F(t) = F_0 \sin(\omega t)$, and solve numerically the driven damped harmonic oscillator (26) in the time interval $t \in [t_0, t_1] = [0, 80]$ s for $\omega_0 = 1$ s^{-1}, $2\zeta\omega_0 = 0.2$ s^{-1}, $F_0/m = 0.1$ m s^{-2}, and $\omega = 1.2$ s^{-1}, and with initial conditions $x_0 = 0.2$ m and $v_0 = 0.8$ m s^{-1}.

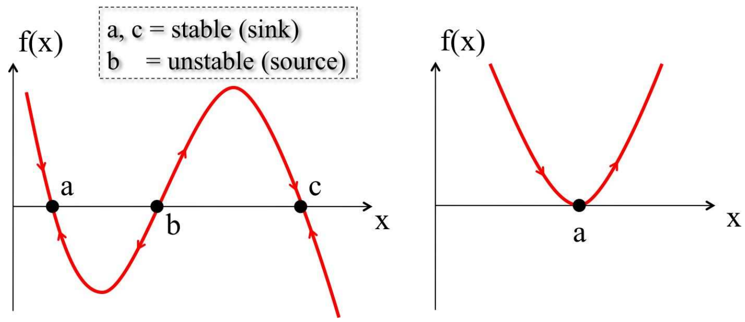

Figure 6: Schematic representation of a phase line diagram and critical points of a differential equation with a form $\frac{dx(t)}{dt} = f(x)$. Left panel: The critical points a and c are stable, while the critical point b is unstable. Right panel: phase line diagram of exercise 9.0.5.

8 Application: Planar Pendulum

8.0.4 Exercise:

Solve numerically the planar pendulum

$$\frac{d^2\theta}{d\tilde{t}^2} = -\sin\theta \; . \tag{27}$$

with initial condition $\theta_0 = 30°$ and $v_0 = 0$ in the interval $\tilde{t} \in [0, 12\pi]$ by making use of both the Euler method and the 2$^{\text{nd}}$–order Runge–Kutta algorithm.

Questions:

1. Plot the numerical solution. Determine for which value of the integration steps N the solution starts not to visibly change any longer;
2. for this value of N, compare by plotting the numerical solution with the exact solution $\theta(\tilde{t}) = \theta_0 \cos(\tilde{t})$ valid for small angles only. Comment the result you get;
3. change the initial condition θ_0: for which value of θ_0 the two curves are approximatively equal? Why?
4. plot the solution in the phase space θ, $d\theta/d\tilde{t}$.

9 Critical points and phase lines

The qualitative behaviour of an ordinary differential equation of the form

$$\frac{dx(t)}{dt} = f(x) \qquad\qquad x(0) = x_0 \; , \tag{28}$$

can be deduced by the phase line diagram (see the schematic Fig. 6), where one can identify the critical points, i.e. the points where $f(x) = 0$, and their stability. In the particular example of the left panel of Fig. 6, the critical points

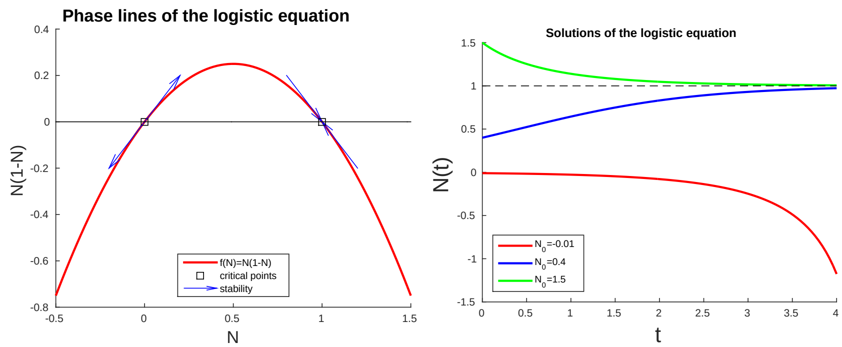

Figure 7: Left: Critical points and phase lines of the logistic equation (30); solutions of the equation for different initial conditions.

a and c are stable and thus act as sinks: all the solutions with initial conditions $x_0 < b$ tend asymptotically to a, while the solutions with $x_0 > b$ tend to c:

$$\lim_{t\to+\infty} x(t) = \begin{cases} a^- & x_0 < a \\ a^+ & a < x_0 < b \end{cases} \qquad \lim_{t\to+\infty} x(t) = \begin{cases} c^- & b < x_0 < c \\ c^+ & x_0 > c \; . \end{cases}$$

9.0.5 Exercise:

Solve both numerically and analytically the differential equation (see right panel of Fig. 6):

$$\frac{dx(t)}{dt} = x^2 \qquad\qquad x(0) = x_0 \; ,$$

with initial conditions $x_0 < 0$ and $x_0 > 0$ and compare the solutions you get.

9.0.6 Exercise:

Find the phase lines, critical point and stability of the differential equation

$$\frac{dx(t)}{dt} = 1 - x(t) \; .$$

Solve it exactly and numerically for $x(0) = x_0 > 1$ and $x_0 < 1$.

9.0.7 Exercise:

Find the phase lines, critical point and stability of the differential equation

$$\frac{dx(t)}{dt} = \log[x(t)] + x(t)^{1/3} - 0.2x(t)^2 \; .$$

Solve it numerically for different initial conditions x_0.

10 Application: Logistic equation

Equations of the form

$$\frac{dN(t)}{dt} = \kappa N(t) \qquad\qquad N(0) = N_0 \; , \tag{29}$$

describe either the exponential growth or the exponential decay of a population at a rate proportional to the size of the population, $N(t)$, with a rate constant κ. If $\kappa > 0$, $N(t)$ grows exponentially, $N(t) = N(0)e^{\kappa t}$, while if $\kappa < 0$, the population decays exponentially, $N(t) = N(0)e^{-|\kappa| t}$.

A more accurate model assumes that the relative growth start decreasing when $N(t)$ approaches a fraction of the carrying capacity N_c of the environment — population growth in a constrained environment. The corresponding equation is called logistic differential equation, and, for $N_c = 1$ and $\kappa = 1$, reads as:

$$\frac{dN(t)}{dt} = N(t)\,[1 - N(t)] \qquad\qquad N(0) = N_0 \; . \tag{30}$$

As explained in the previous section, there is no need to explicitly solve this equation to understand the behaviour of the population $N(t)$ with the time t for different initial conditions N_0, rather one has to understand the phase lines and the stability of the two critical points $N = 0$ and $N = 1$ — see left panel of Fig. 7. Nevertheless, Eq. (30) can be solved exactly by variable separation, giving:

$$N(t) = \frac{N_0}{N_0 + (1 - N_0)e^{-t}} \; . \tag{31}$$

Different solutions can be seen in the right panel of Fig. 7.

10.0.8 Exercise:

Numerically solve the logistic equation (30) for different values of the initial population N_0 and compare the numerical result you get with the exact one of Eq. (31). Linearise the problem around the critical points $N = 0$ (unstable) and $N = 1$ (stable), and compare the solutions you get in terms of an exponential function with the general solution. When can you use the linearised ones?

10.0.9 Exercise:

Consider the following modified logistic equation

$$\frac{dN(t)}{dt} = aN^2(t)\,[1 - bN(t)] \qquad\qquad N(0) = N_0 \; .$$

Where $a = 2$ and $b = 50$. Find the critical points and their stability. Study the numerical solutions for different initial populations N_0, as well as close to the stable critical point, comparing your results with the ones of the usual logistic equation.

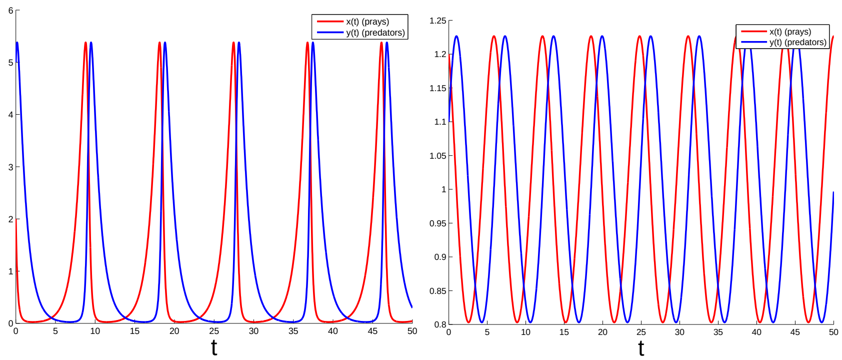

Figure 8: Solutions of the prays and predators problem (33) with parameters $a = b = c = d = 1$ and initial conditions: $x_0 = 1.2$ and $y_0 = 1.1$ (left panel); $x_0 = 2$ and $y_0 = 5$ (right panel).

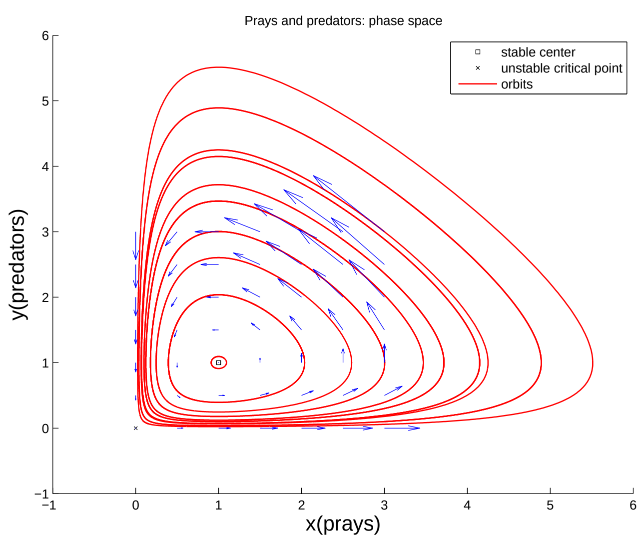

Figure 9: Orbits in the phase space of the pray and predator problems for the following parameters $a = b = c = d = 1$ and different initial conditions x_0 and y_0.

11 Application: Prays and predators

Volterra-Lotka equations are differential equations that can be used to model predator-prey interactions. The original system discovered by both Volterra and Lotka independently consisted of two entities. Vito Volterra developed these equations in order to model a situation where one type of fish is the prey for another type of fish. The model was simplified by the following assumptions:

1. The prey population increases exponentially in the absence of predators.
2. The predator population decreases exponentially in the absence of prey.
3. The prey population decreases relative to the frequency with which predators meet prey as a result of predation.
4. The predator population increases relative to the frequency with which predators meet prey as a result of predation.

Using these assumptions, the Volterra-Lotka equations for the two-dimensional predator-prey system with exponential growth is defined by the following system of differential equations:

$$\frac{dx(t)}{dt} = ax - bxy \qquad x(0) = x_0 \tag{32}$$

$$\frac{dy(t)}{dt} = dxy - cy \qquad y(0) = y_0 \ . \tag{33}$$

The critical points, $\frac{dx(t)}{dt} = 0 = \frac{dy(t)}{dt}$ are given by $(x,y) = (0,0)$ and $(x,y) = (\frac{a}{b}, \frac{c}{d})$. By evaluating the Jacobian

$$J(x,y) = \begin{pmatrix} a - by & -bx \\ dy & dx - c \end{pmatrix} \tag{34}$$

it can shown that the first is unstable, while the second is a stable center — see Fig. 9.

11.0.10 Exercise:

Numerically solve the prays and predators equations (33) for $a = 4$, $b = 2$, $c = 3$, and $d = 3$ and for different initial conditions x_0 and y_0 far and close to the critical points. Plot the corresponding orbits as in Fig. 9 — plot also the flow lines. Linearise the equations close to the stable critical point, evaluate analytically the solutions and plot them together with the numerical ones for the full problem.

11.0.11 Exercise:

Numerically solve the following equations describing two competing populations

$$\frac{dx(t)}{dt} = 60x - 4x^2 - 3xy$$
$$\frac{dy(t)}{dt} = 42y - 2y^2 - 3xy$$

for different initial conditions x_0 and y_0. Evaluate the critical points and their stability. Linearise close to one of the stable critical points, evaluate analytically the solutions and plot them together with the numerical ones for the full problem.