Problem set 4

F. M. Marchetti[*]

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I. ODLRO IN A WEAKLY INTERACTING BOSE GAS

The one-body density matrix can be defined for an interacting systems as the following expectation value of field operators:

$$n^{(1)}(\mathbf{r}_1, \mathbf{r}_2) = \langle \hat{\Psi}^\dagger(\mathbf{r}_1) \hat{\Psi}(\mathbf{r}_2) \rangle \; . \tag{1}$$

The field operator $\hat{\Psi}(\mathbf{r})$ represents the probability amplitude of destroying a particle at the position $\mathbf{r}$ and, for a homogeneous gas in a volume V of interacting bosons can be written in terms of the destruction operator $\hat{a}_\mathbf{p}$ of a particle in the momentum state $\mathbf{p}$:

$$\hat{\Psi}(\mathbf{r}) = \sum_\mathbf{p} \frac{e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}}{\sqrt{V}} \hat{a}_\mathbf{p} \; , \tag{2}$$

Thus, for a translationally invariant system, the one-body density matrix is a function of $s = |\mathbf{s}| = |\mathbf{r}_1 - \mathbf{r}_2|$, as it should be.

1. Considering that, for $T \ll T_c$, the interacting Hamiltonian

   $$\hat{H} \simeq \frac{U_0 N^2}{2V} + \sum_{\mathbf{p}\neq 0} E_\mathbf{p} \hat{b}^\dagger_\mathbf{p} \hat{b}_\mathbf{p} + \frac{1}{2} \sum_{\mathbf{p}\neq 0} (E_\mathbf{p} - \epsilon_\mathbf{p} - U_0 n) \; . \tag{3}$$

   becomes diagonal in the basis of quasiparticle operators

   $$\begin{pmatrix} \hat{b}_\mathbf{p} \\ \hat{b}^\dagger_{-\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \cosh\theta_\mathbf{p} & -\sinh\theta_\mathbf{p} \\ -\sinh\theta_\mathbf{p} & \cosh\theta_\mathbf{p} \end{pmatrix} \begin{pmatrix} \hat{a}_\mathbf{p} \\ \hat{a}^\dagger_{-\mathbf{p}} \end{pmatrix} \; , \tag{4}$$

   demonstrate that

   $$n^{(1)}(s) = n_0 + \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} e^{-i\mathbf{p}\cdot\mathbf{s}/\hbar} n_\mathbf{p} \; , \tag{5}$$

   where n_0 is the (depleted) condensate density, and where the particle momentum distribution is given by

   $$n_\mathbf{p} = \langle \hat{a}^\dagger_\mathbf{p} \hat{a}_\mathbf{p} \rangle = \sinh^2\theta_\mathbf{p} (1 + \langle \hat{b}^\dagger_{-\mathbf{p}} \hat{b}_{-\mathbf{p}} \rangle) + \cosh^2\theta_\mathbf{p} \langle \hat{b}^\dagger_\mathbf{p} \hat{b}_\mathbf{p} \rangle \; . \tag{6}$$

2. Find the implicit expression of $n^{(1)}(s)$ both for $T \ll T_c$ and $T = 0$.
3. Demonstrate that, at $T = 0$,

   $$n^{(1)}(s) = n_0 + \frac{1}{4\pi s \xi^2} \int_0^\infty dx x \sin(xs/\xi) \left(\frac{x^2+1}{\sqrt{x^2(x^2+2)}} - 1 \right) \; , \tag{7}$$

   where $\xi = \frac{\hbar}{\sqrt{2mU_0 n}}$ is the healing length.
4. Plot numerically $n^{(1)}(s)$ and show analytically that $\lim_{s\gg\xi} n^{(1)}(s) = n_0$.

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[*] Electronic address: `francesca.marchetti@uam.es`

II. LANDAU CRITERION

According to the Landau criterion for superfluidity, quasi-particles can be excited in a fluid where a small defect is moving at a constant velocity $\mathbf{v} = (v, 0)$ (let's consider the 2D case here) if the condition

$$E'_{\mathbf{p}} \equiv E_p - \mathbf{p} \cdot \mathbf{v} = \sqrt{\frac{U_0 n p^2}{m} + (\frac{p^2}{2m})^2} - \mathbf{p} \cdot \mathbf{v} < 0 \tag{8}$$

is satisfied.

5. Show that the defect critical velocity for quasi-particles excitation with a Bogoliubov dispersion $E_p = \sqrt{\frac{U_0 n p^2}{m} + (\frac{p^2}{2m})^2}$ is given by the speed of sound

$$v_c = \min_{\mathbf{p}} \frac{E_p}{p} = c_s = \sqrt{U_0 n / m}\ . \tag{9}$$

6. In 2D ($p^2 = p_x^2 + p_y^2$) find the closed curve Γ in the (p_x, p_y)-plane for which $E'_{\mathbf{p}} = 0$ is satisfied and plot it — N.B. the curve reduces to a point if $v \leq v_c = c_s$.

The spectrum of collective excitations of superfluid ^{4}He is different from that of a weakly interacting Bose-Einstein condensate, because it is not only characterised by a phonon-like behaviour at low momenta and energies, but also a roton minimum for some intermediated momenta at which the spectrum has a gap. Overall, the roton spectrum properties are thus:

$$E_p \simeq \begin{cases} c_s p & p \to 0 \\ \Delta + \dfrac{(p - p_r)^2}{2m} & p \simeq p_r\ , \end{cases} \tag{10}$$

where Δ is the roton gap and p_r the roton minimum.

7. Consider the case of a gapped spectrum $E_p = \Delta + \frac{(p-p_r)^2}{2m}$ and evaluate the critical velocity v_c.
8. Let us consider the following spectrum which interpolates between the two behaviours above and a free spectrum at large momenta:

$$\tilde{E}_{\tilde{p}} = \sqrt{\frac{\tilde{p}^2}{2}\left(\frac{\tilde{p}^2}{2} + \alpha(1 - \tilde{p})\right)}\ . \tag{11}$$

Here we use a dimensionless notation where energy and momenta are expressed in terms of the scales E_s and p_s, respectively, i.e., $\tilde{E} = E/E_s$ and $\tilde{p} = p/p_s$.

9. Demonstrate that the spectrum $\tilde{E}_{\tilde{p}}$ admits a roton minimum for the parameter $\alpha \in [16/9, 2)]$
10. Evaluate the roton minimum $\tilde{p}_r$ and demonstrate that

$$\tilde{p}_r = \frac{3\alpha + \sqrt{\alpha(9\alpha - 16)}}{4}\ . \tag{12}$$

11. Evaluate the roton gap $\tilde{\Delta}$ and plot it as a function of the parameter $\alpha \in [16/9, 2)]$.
12. Demonstrate the following asymptotic behaviours

$$\tilde{E}_{\tilde{p}} \simeq \begin{cases} \sqrt{\dfrac{\alpha}{2}}\tilde{p} & \tilde{p} \to 0 \\ \tilde{\Delta} + \beta(\tilde{p} - \tilde{p}_r)^2 & \tilde{p} \simeq \tilde{p}_r \\ \dfrac{1}{2}\tilde{p}^2 - \dfrac{\alpha}{2}\tilde{p} - \dfrac{\alpha}{4}(\alpha - 2) & \tilde{p} \to \infty\ . \end{cases} \tag{13}$$

13. Apply the Landau criterion for superfluidity and evaluate the dimensionless critical velocity and show that:

$$\tilde{v}_c = \min_{\tilde{p}} \left(\frac{\tilde{E}_{\tilde{p}}}{\tilde{p}}\right) = \frac{\sqrt{\alpha(2 - \alpha)}}{2}\ . \tag{14}$$