“Condensed Matter Physics” course (part 2): Problem set 2

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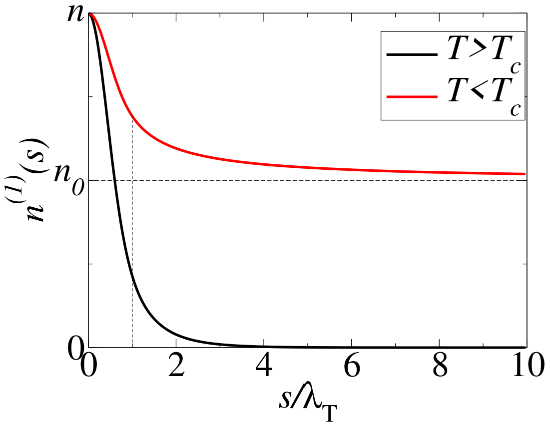
I. CANONICAL TRANSFORMATIONS

1. Consider the following two-state Hamiltonian
$$\hat{H} = \begin{pmatrix} \hat{a}_1^\dagger & \hat{a}_2 \end{pmatrix} \begin{pmatrix} \epsilon & \Delta \\ \Delta & \epsilon \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} , \tag{1}$$
for two **fermionic** fields (all operators anti-commute except $\{\hat{a}_1, \hat{a}_1^\dagger\} = 1 = \{\hat{a}_2, \hat{a}_2^\dagger\}$). Find the transformation to a new set of operators
$$\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2^\dagger \end{pmatrix} = O \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} \tag{2}$$
where O is a 2×2 matrix, such that the fermionic canonical transformation relations are preserved for $\hat{\gamma}_i$ and that simultaneously diagonalises $\hat{H}$ to the form:
$$\hat{H} = E_1 \hat{\gamma}_1^\dagger \hat{\gamma}_1 + E_2 \hat{\gamma}_2^\dagger \hat{\gamma}_2 . \tag{3}$$
Find the expression for the energies $E_{1,2}$ in terms of ϵ and Δ.
2. Comments about the different form the matrix O has in the case where the operators $\hat{a}_1$ and $\hat{a}_2$ describe bosonic particles and compare the expressions of the energies $E_{1,2}$ in the fermionic and bosonic cases.

II. ONE-BODY DENSITY MATRIX AND ODLRO (NUMERICAL)

Starting from the definition of the one-body density matrix for the case of an ideal gas in a 3D box,
$$n^{(1)}(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{p}} f_{\mathbf{p}} \varphi_{\mathbf{p}}^*(\mathbf{r}) \varphi_{\mathbf{p}}(\mathbf{r}') , \tag{4}$$
where the eigenfunctions of $\widehat{\mathcal{H}} = \frac{\hat{\mathbf{p}}}{2m}$ are plane waves, $\varphi_{\mathbf{p}}(\mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}/\sqrt{V}$, and where the Bose-Einstein distribution in the grand-canonical ensemble is given by $f_{\mathbf{p}} = \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\mu)}-1}$,

3. evaluate numerically $n^{(1)}(s)$ as a function of the distance $s = |\mathbf{r} - \mathbf{r}'|$ in units of the de Broglie wavelength λ_T.
4. Check that your numerical results coincides with the analytical behaviours obtained in class. In particular, check that at large distances ($s = |\mathbf{r} - \mathbf{r}'| \gg \lambda_T$) $n^{(1)}(s) \simeq n_0 + 1/(\lambda_T^2 s)$ for $T < T_c$, where $n_0 = N_0/V$ is the condensate density, while $n^{(1)}(s) \simeq e^{\beta\mu} e^{-\sqrt{4\pi(1-e^{\beta\mu})} s/\lambda_T}/(\lambda_T^2 s)$ for $T > T_c$.
5. Consider the case of a classical gas, where the distribution function is the Maxwell-Boltzmann one, $f_{\mathbf{p}} = e^{-\beta\epsilon_{\mathbf{p}}}$ and evaluate the one-body density matrix $n^{(1)}(s)$. Show that if you consider the short-distance behaviour $s \ll \lambda_T$ of the general quantum case, you get exactly the same result.



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III. COHERENT STATES

We want to find the state $|\psi\rangle$ that is an eigenstate of the destruction operator $\hat{a}$,

$$\hat{a}|\psi\rangle = \psi|\psi\rangle \ , \tag{5}$$

and express it in terms of the number state

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \ . \tag{6}$$

6. Assume that

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \ , \tag{7}$$

and find the expression of the coefficients c_n that satisfy (5), as well as the normalisation condition $\langle\psi|\psi\rangle = 1$, demonstrating that

$$|\psi\rangle = e^{-\frac{|\psi|^2}{2}} e^{\psi\hat{a}^\dagger}|0\rangle \ . \tag{8}$$

7. Further demonstrate that you can also write the coherent state $|\psi\rangle$ in the following equivalent form

$$|\psi\rangle = e^{\psi\hat{a}^\dagger - \psi^*\hat{a}}|0\rangle \ . \tag{9}$$

8. Evaluate the uncertainty in the number of particles (i.e., the variance) in a coherent state

$$\Delta N = \sqrt{\langle\psi|\hat{N}^2|\psi\rangle - \langle\psi|\hat{N}|\psi\rangle^2} \ , \tag{10}$$

and show that $\lim_{N\to\infty} \Delta N/N = 0$.

IV. GROUND STATE OF A WEAKLY INTERACTING BOSE GAS

We have seen in class that, considering the Bogoliubov approximation and the following expansion (mean-field+quantum-fluctuations), we can get to the following Hamiltonian

$$\hat{H} \simeq \frac{U_0 N^2}{2V} + \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \frac{U_0 n}{2}\sum_{\mathbf{p}\neq 0}\left(2\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}}\right) \ . \tag{11}$$

In addition, we have shown in class that we can diagonalise the Hamiltonian (11) via the Bogoliubov transformation:

$$\begin{pmatrix} \hat{b}_{\mathbf{p}} \\ \hat{b}^\dagger_{-\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \cosh\theta_{\mathbf{p}} & -\sinh\theta_{\mathbf{p}} \\ -\sinh\theta_{\mathbf{p}} & \cosh\theta_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} \hat{a}_{\mathbf{p}} \\ \hat{a}^\dagger_{-\mathbf{p}} \end{pmatrix} \ , \tag{12}$$

getting to the following diagonal form in terms of the quasi-particle operators

$$\hat{H} \simeq \frac{U_0 N^2}{2V} + \sum_{\mathbf{p}\neq 0} E_{\mathbf{p}} \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}} + \frac{1}{2}\sum_{\mathbf{p}\neq 0}\left(E_{\mathbf{p}} - \epsilon_{\mathbf{p}} - U_0 n\right) \ . \tag{13}$$

where

$$\sinh 2\theta_{\mathbf{p}} = \frac{-U_0 n}{E_{\mathbf{p}}} \qquad \cosh 2\theta_{\mathbf{p}} = \frac{\epsilon_{\mathbf{p}} + U_0 n}{E_{\mathbf{p}}}$$

$$\sinh\theta_{\mathbf{p}} = -\sqrt{\frac{1}{2}\left(\frac{\epsilon_{\mathbf{p}} + U_0 n}{E_{\mathbf{p}}} - 1\right)} \qquad \cosh\theta_{\mathbf{p}} = \sqrt{\frac{1}{2}\left(\frac{\epsilon_{\mathbf{p}} + U_0 n}{E_{\mathbf{p}}} + 1\right)} \ ,$$

and where the spectrum of the quasi-particles excitations is given by $E_{\mathbf{p}} = \sqrt{\epsilon_{\mathbf{p}} \left(\epsilon_{\mathbf{p}} + 2U_0 n\right)}$.

At the mean-field level, we discussed in class that the ground state of a BEC at zero temperature $T = 0$ is a coherent state. We want to demonstrate now that, when including interactions, the ground state of the weakly interacting Bose gas at $T = 0$ is modified by the occupation of the excited states (because of the interaction depleting the condensate) and it is given by

$$|\tilde{\psi}_0\rangle = \mathcal{N} e^{\psi_0 \hat{a}_0^\dagger + \frac{1}{2}\sum_{\mathbf{p}\neq 0} \tanh\theta_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{-\mathbf{p}}^\dagger} |0\rangle \;, \tag{14}$$

where $\mathcal{N}$ is a normalisation factor guaranteeing that $\langle\tilde{\psi}_0|\tilde{\psi}_0\rangle = 1$.

9. Check that

$$\hat{b}_{\mathbf{p}} |\tilde{\psi}_0\rangle = 0 \;. \tag{15}$$

10. Consider the following simplified problem of a bosonic two-state Hamiltonian

$$\hat{H} = \begin{pmatrix} \hat{a}_1^\dagger & \hat{a}_2 \end{pmatrix} \begin{pmatrix} \epsilon & \Delta \\ \Delta & \epsilon \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} \;. \tag{16}$$

We have previously demonstrated how this Hamiltonian can be diagonalised using the Bogoliubov transformation

$$\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} \;. \tag{17}$$

11. Demonstrate that $\hat{\gamma}_i|\text{g.s.}\rangle = 0$ for

$$|\text{g.s.}\rangle = \mathcal{N} e^{\tanh\theta \hat{a}_1^\dagger \hat{a}_2^\dagger} |0\rangle \;. \tag{18}$$

12. Further, find the normalisation factor $\mathcal{N}$, by requiring that $\langle \text{g.s.}|\text{g.s.}\rangle = 1$.