

“Condensed Matter Physics” course (part 2): Problem set 3

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(Dated: April 16, 2020)

I. ODLRO IN A WEAKLY INTERACTING BOSE GAS

The one-body density matrix can be defined for an interacting systems as the following expectation value of field operators:

$$n^{(1)}(\mathbf{r}_1, \mathbf{r}_2) = \langle \hat{\Psi}^\dagger(\mathbf{r}_1) \hat{\Psi}(\mathbf{r}_2) \rangle . \quad (1)$$

The field operator $\hat{\Psi}(\mathbf{r})$ represents the probability amplitude of destroying a particle at the position \mathbf{r} and, for a homogeneous gas in a volume V of interacting bosons can be written in terms of the destruction operator $\hat{a}_{\mathbf{p}}$ of a particle in the momentum state \mathbf{p} :

$$\hat{\Psi}(\mathbf{r}) = \sum_{\mathbf{p}} \frac{e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}}{\sqrt{V}} \hat{a}_{\mathbf{p}} . \quad (2)$$

Thus, for a translationally invariant system, the one-body density matrix is a function of $s = |\mathbf{s}| = |\mathbf{r}_1 - \mathbf{r}_2|$.

1. Considering that, for $T \ll T_c$, the interacting Hamiltonian

$$\hat{H} \simeq \frac{U_0 N^2}{2V} + \sum_{\mathbf{p} \neq 0} E_{\mathbf{p}} \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p} \neq 0} (E_{\mathbf{p}} - \epsilon_{\mathbf{p}} - U_0 n) . \quad (3)$$

becomes diagonal in the basis of quasiparticle operators

$$\begin{pmatrix} \hat{b}_{\mathbf{p}} \\ \hat{b}_{-\mathbf{p}}^\dagger \end{pmatrix} = \begin{pmatrix} \cosh \theta_{\mathbf{p}} & -\sinh \theta_{\mathbf{p}} \\ -\sinh \theta_{\mathbf{p}} & \cosh \theta_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} \hat{a}_{\mathbf{p}} \\ \hat{a}_{-\mathbf{p}}^\dagger \end{pmatrix} , \quad (4)$$

demonstrate that

$$n^{(1)}(s) = n_0 + \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} e^{-i\mathbf{p}\cdot\mathbf{s}/\hbar} n_{\mathbf{p}} , \quad (5)$$

where n_0 is the (depleted) condensate density, and where the particle momentum distribution is given by

$$n_{\mathbf{p}} = \langle \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} \rangle = \sinh^2 \theta_{\mathbf{p}} (1 + \langle \hat{b}_{-\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}} \rangle) + \cosh^2 \theta_{\mathbf{p}} \langle \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \rangle . \quad (6)$$

2. Find the implicit expression of $n^{(1)}(s)$ both for $T \ll T_c$ and $T = 0$.
3. Demonstrate that, at $T = 0$,

$$n^{(1)}(s) = n_0 + \frac{1}{4\pi^2 s \xi^2} \int_0^\infty dx x \sin\left(\frac{xs}{\xi}\right) \left[\frac{1+x^2}{\sqrt{2x^2+x^4}} - 1 \right] , \quad (7)$$

where $\xi = \frac{\hbar}{\sqrt{2mU_0n}}$ is the healing length.

4. Plot $n^{(1)}(s)$ numerically and show analytically that $\lim_{s \gg \xi} n^{(1)}(s) = n_0$.

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II. LANDAU CRITERION

According to the Landau criterion for superfluidity, quasi-particles can be excited in a fluid where a small defect is moving at a constant velocity $\mathbf{v} = (v, 0)$ (let's consider the 2D case here) if the condition

$$E'_{\mathbf{p}} \equiv E_p - \mathbf{p} \cdot \mathbf{v} = \sqrt{\frac{U_0 n p^2}{m} + \left(\frac{p^2}{2m}\right)^2} - \mathbf{p} \cdot \mathbf{v} < 0 \quad (8)$$

is satisfied.

5. Show that the defect critical velocity for quasi-particles excitation with a Bogoliubov dispersion $E_p = \sqrt{\frac{U_0 n p^2}{m} + \left(\frac{p^2}{2m}\right)^2}$ is given by the speed of sound

$$v_c = \min_{\mathbf{p}} \frac{E_p}{p} = c_s = \sqrt{U_0 n / m} . \quad (9)$$

The spectrum of collective excitations of superfluid ^4He is different from that of a weakly interacting Bose-Einstein condensate, because it is not only characterised by a phonon-like behaviour at low momenta and energies, but also a roton minimum for some intermediated momenta at which the spectrum has a gap. Overall, the roton spectrum properties are thus:

$$E_p \simeq \begin{cases} c_s p & p \rightarrow 0 \\ \Delta + \frac{(p - p_r)^2}{2m} & p \simeq p_r , \end{cases} \quad (10)$$

where Δ is the roton gap and p_r the roton minimum.

6. Consider the case of a gapped spectrum $E_p = \Delta + \frac{(p - p_r)^2}{2m}$ and evaluate the critical velocity v_c .
7. Let us consider the following spectrum which interpolates between the two behaviours above and a free spectrum at large momenta:

$$\tilde{E}_{\tilde{p}} = \sqrt{\frac{\tilde{p}^2}{2} \left(\frac{\tilde{p}^2}{2} + \alpha(1 - \tilde{p}) \right)} . \quad (11)$$

Here we use a dimensionless notation where energy and momenta are expressed in terms of the scales E_s and p_s , respectively, i.e., $\tilde{E} = E/E_s$ and $\tilde{p} = p/p_s$.

8. Demonstrate that the spectrum $\tilde{E}_{\tilde{p}}$ admits a roton minimum for the parameter $\alpha \in [16/9, 2]$
9. Evaluate the roton minimum \tilde{p}_r and demonstrate that

$$\tilde{p}_r = \frac{3\alpha + \sqrt{\alpha(9\alpha - 16)}}{4} . \quad (12)$$

10. Evaluate the roton gap $\tilde{\Delta}$ and plot it as a function of the parameter $\alpha \in [16/9, 2]$.

11. Demonstrate the following asymptotic behaviours

$$\tilde{E}_{\tilde{p}} \simeq \begin{cases} \sqrt{\frac{\alpha}{2}} \tilde{p} & \tilde{p} \rightarrow 0 \\ \tilde{\Delta} + \beta(\tilde{p} - \tilde{p}_r)^2 & \tilde{p} \simeq \tilde{p}_r \\ \frac{1}{2}\tilde{p}^2 - \frac{\alpha}{2}\tilde{p} - \frac{\alpha}{4}(\alpha - 2) & \tilde{p} \rightarrow \infty . \end{cases} \quad (13)$$

12. Apply the Landau criterion for superfluidity and evaluate the dimensionless critical velocity and show that:

$$\tilde{v}_c = \min_{\tilde{p}} \left(\frac{\tilde{E}_{\tilde{p}}}{\tilde{p}} \right) = \frac{\sqrt{\alpha(2 - \alpha)}}{2} . \quad (14)$$

III. THE HOMOGENEOUS IDEAL FERMI GAS IN 3D AND IN 2D. PART I: ANALYTICS

Consider first a homogeneous gas of N free fermions in a single spin state (no spin degeneracy) in a 3D volume V .

13. Evaluate the Fermi energy

$$\varepsilon_F = \frac{\hbar^2}{2m} (6\pi^2 n)^{2/3} \quad (15)$$

as a function of the gas density, $n = N/V$, and the particle mass m .

14. Show that the ground state energy of the system at zero temperature is given by $E(T=0) = \frac{3}{5} N \varepsilon_F$ and evaluate the gas compressibility $\kappa^{-1} = -V \frac{\partial P}{\partial V}$, where the pressure is related to the internal energy by $PV = \frac{2}{3} E$. Compare and comment the result you get with the one you got for an ideal gas of bosons.
15. For a polarised gas of fermionic particles, demonstrate the validity of following expansions at low temperature ($T \ll T_F$) for the chemical potential and the system internal energy, respectively:

$$\begin{aligned} \mu(T) &\simeq \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{T}{T_F} \right)^2 + \dots \right] \\ E(T) &\simeq \frac{3}{5} N \varepsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{T}{T_F} \right)^2 \right]. \end{aligned}$$

16. Consider now a homogeneous gas of identical fermions (single spin state) in 2D: evaluate the Fermi energy and, at finite temperature, establish that:

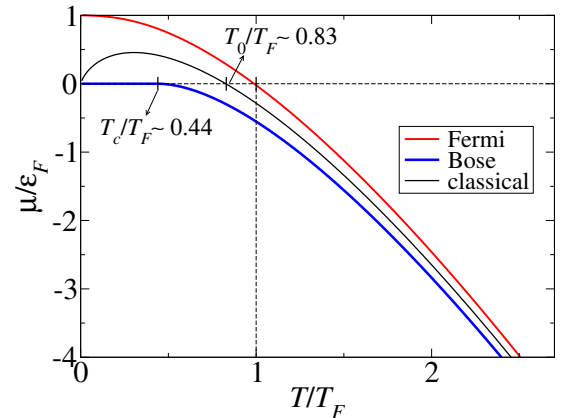
$$\varepsilon_F = \mu + k_B T \log(1 + e^{-\beta\mu}). \quad (16)$$

Note that in 2D the density of state is constant, and the integral fixing the number of particles can be evaluated analytically. Invert this expression and plot $\mu/k_B T_F$ versus T/T_F together with the result you get for the chemical potential of a classical gas in 2D. What happens for a homogeneous gas of bosons in 2D?

IV. THE HOMOGENEOUS IDEAL FERMI GAS IN 3D. PART II: NUMERICS

Consider a homogeneous gas of N free fermions in a single spin state (no spin degeneracy) in a 3D volume V .

17. Evaluate numerically the expression of the chemical potential $\mu(T)/k_B T_F$ as a function of the rescaled temperature T/T_F , where the Fermi temperature is given by $T_F = k_B \varepsilon_F$. Compare the plot you get with the case of a) a classical gas of single component particles (here you can do the calculation analytically), where you define the temperature scale $k_B T_0 = \frac{2\pi\hbar^2}{m} n^{2/3}$ from the condition $n\lambda_{T_0}^3 = 1$, and with the case of b) a gas of identical bosons, where the critical temperature for BEC is given by $k_B T_c = \frac{2\pi\hbar^2}{m} n^{2/3} / [g_{3/2}(1)]^{2/3}$ (you got the plot by solving problem set 1). Your result should look like the one shown in the figure here. Comment the results you get. Comment about the fact that the critical temperature for BEC is the same order of magnitude of the Fermi energy, $T_c \sim T_F (\sim T_0)$ even though T_c signals the onset of a phase transition while T_F indicates a crossover from the classical to the quantum regime.



18. Plot the system internal energy $E(T)$ and heat capacity $c_v = \frac{\partial E(T)}{\partial T}$ as a function of the rescaled temperature T/T_F for fermions, bosons and a classical particles.