Problem set 1

January 30, 2013

to submit by Monday the 11^{th} of February

1 Ideal Bose and Fermi gases

Starting from the expression of the partition function

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta(\hat{H}-\mu\hat{N})}] = \prod_{i=0}^{\infty} \sum_{N_i} e^{-\beta(\epsilon_i-\mu)N_i} , \tag{1}$$

in the grand canonical ensemble,

1. check that the mean number of particles N and energy E, are given (for both bosons $[N_i = 0, 1, 2, 3, \dots]$ and fermions $[N_i = 0, 1]$) by

$$N = \sum_{i=0}^{\infty} f_i^{B,F} \qquad\qquad E = \sum_{i=0}^{\infty} \epsilon_i f_i^{B,F} , \tag{2}$$

where the mean occupation number of the i-th energy level is given by the Bose-Einstein and the Fermi-Dirac distribution respectively, $f_i^{B,F} = 1/(e^{\beta(\epsilon_i-\mu)} \mp 1)$.

2 Ideal Bose gas in a 3D box

Consider an ideal Bose gas of N (non-interacting) bosons in a three-dimensional (3D) cubic box with periodic boundary conditions. The gas is at thermal equilibrium at a temperature T.

2. Show that the expression for the BEC critical temperature

$$k_B T_c = \frac{2\pi}{[g_{3/2}(1)]^{2/3}} \frac{\hbar^2 n^{2/3}}{m} \simeq 3.31 \frac{\hbar^2 n^{2/3}}{m} \tag{3}$$

is dimensionally correct. Then estimate the critical temperature for condensation T_c on the basis of this expression for a density $n \simeq 10^{13} - 10^{15}$ cm^{-3} and the mass of ^{87}Rb atoms ($m = 87$ u, where u is the atomic mass unit).

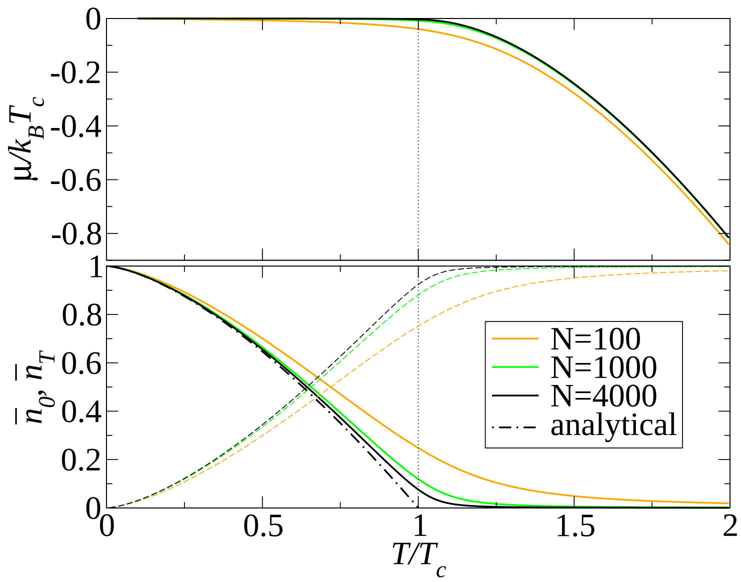
3. Invert numerically the equation which fixes the total number of particles N,

$$N = \frac{1}{e^{-\beta\mu} - 1} + N \left(\frac{T}{T_c}\right)^{3/2} \frac{\int_0^\infty dx \frac{\sqrt{x}}{e^{-\beta\mu} e^x - 1}}{\int_0^\infty dx \frac{\sqrt{x}}{e^x - 1}} , \tag{4}$$

in order to obtain the rescaled chemical potential $\mu/(k_B T_c)$ as a function of N and the rescaled temperature T/T_c, where T_c is the critical temperature for BEC.

4. Plot $\mu/(k_B T_c)$ as a function of T/T_c for different values of the total number of particles N and comment the results you get.

5. Plot the condensate fraction $\bar{n}_0 = N_0(N, T/T_c)/N$ and the thermal fraction $\bar{n}_T = N_T(N, T/T_c)/N$ as a function of T/T_c for different values of the total number of particles N and compare the numerical results with the analytical formula valid in the thermodynamic limit.



3 Ideal Bose gas in 2D

6. Show that for a free gas in a box of dimension $d = 1, 2, 3$ and volume $V = L^d$, one has in general that the DoS is given by

$$\mathcal{N}(\epsilon) = \frac{dG(\epsilon)}{d\epsilon} = \frac{\Omega_d}{2} \frac{V}{(2\pi)^d} \left(\frac{2m}{\hbar^2}\right)^{d/2} \epsilon^{d/2-1} \tag{5}$$

$$G(\epsilon) = \sum_{\mathbf{p}}^{\epsilon_{\mathbf{p}} < \epsilon} , \tag{6}$$

where $\Omega_d = 4\pi (d = 3), 2\pi (d = 2), 1 (d = 1)$.

7. Evaluate the condition to find the critical temperature of an ideal gas in a 2D box (with boundary conditions) and comment why the condition

$N_T(\mu \to 0^-, T) \leq N$ can never be satisfied at finite temperature. Can you find a critical temperature T_c?

8. Do now the same calculation for an ideal gas embedded in a 2D harmonic trap. Evaluate the critical temperature.