Problem set 2

February 8, 2013

to submit by Tuesday the 19^{th} of February

1 One-body density matrix and OLRO

Starting from the definition of the one-body density matrix for the case of an ideal gas in a 3D box,

$$n^{(1)}(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{p}} f_{\mathbf{p}} \varphi^*_{\mathbf{p}}(\mathbf{r}) \varphi_{\mathbf{p}}(\mathbf{r}') , \tag{1}$$

where the eigenfunctions of $\widehat{\mathcal{H}} = \frac{\widehat{\mathbf{p}}}{2m}$ are plane waves, $\varphi_{\mathbf{p}}(\mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}/\sqrt{V}$, and $f_{\mathbf{p}} = \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\mu)}-1}$,

1. evaluate analytically the behaviour at large distances ($s = |\mathbf{r} - \mathbf{r}'| \gg \lambda_T$) of $n^{(1)}(s)$ for $T < T_c$, and show that $n^{(1)}(s) \simeq n_0 + 1/(\lambda_T^2 s)$, where λ_T is the de Broglie wavelength and $n_0 = N_0/V$ the condensate density.
2. Repeat the same calculation for $T > T_c$ and show that now the decay to zero is of the type of Yukawa-law, i.e. $n^{(1)}(s) \simeq e^{\beta\mu} e^{-\sqrt{4\pi(1-e^{\beta\mu})}s/\lambda_T}/(\lambda_T^2 s)$.
3. Consider the case of a classical gas, where the distribution function is the Maxwell-Boltzmann one, $f_{\mathbf{p}} = e^{-\beta\epsilon_{\mathbf{p}}}$ and evaluate the one-body density matrix $n^{(1)}(s)$. Show that if you consider the short-distance behaviour $s \ll \lambda_T$ of the general quantum case, you get exactly the same result.

2 Second quantisation

Consider the definitions of creation and annihilation operators for bosons (occupation $N_i = 0, 1, 2, \dots$)

$$\hat{a}_j^\dagger |N_0, N_1, \dots N_j, \dots\rangle = \sqrt{N_j + 1} |N_0, N_1, \dots, N_j + 1, \dots\rangle \tag{2}$$

$$\hat{a}_j |N_0, N_1, \dots N_j, \dots\rangle = \sqrt{N_j} |N_0, N_1, \dots N_j - 1, \dots\rangle , \tag{3}$$

and for fermions (occupation $N_i = 0, 1$):

$$\hat{c}_j^\dagger |N_0, N_1, \dots N_j, \dots\rangle = (-1)^{\mathcal{P}} (1 - N_j) |N_0, N_1, \dots, N_j + 1, \dots\rangle \tag{4}$$

$$\hat{c}_j |N_0, N_1, \dots N_j, \dots\rangle = (-1)^{\mathcal{P}} N_j |N_0, N_1, \dots N_j - 1, \dots\rangle , \tag{5}$$

where $\mathcal{P} = N_0 + N_1 + \cdots + N_{i-1}$ and where we indicate with $i = 0, 1, \ldots$ the energy levels of the single-particle problem $\widehat{\mathcal{H}}|\epsilon_i\rangle = \epsilon_i|\epsilon_i\rangle$ and with N_i the occupation number of each such a state.

4. Derive the canonical commutation and anticommuation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \qquad [\hat{a}_i, \hat{a}_j] = 0 = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] \; , \tag{6}$$

and

$$\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{ij} \qquad \{\hat{c}_i, \hat{c}_j\} = 0 = \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} \; . \tag{7}$$

5. For bosons show that in general one has that

$$[a_i, \hat{a}_j^\dagger \hat{a}_k] = \delta_{1j} a_k \qquad [a_i^\dagger, \hat{a}_j^\dagger \hat{a}_k] = -\delta_{ik} a_j^\dagger \; . \tag{8}$$

6. Consider the following two-state Hamiltonian

$$\hat{H} = \epsilon \left(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2\right) + \Delta \left(\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_2 \hat{a}_1\right) \; , \tag{9}$$

for two boson fields (all operators commute except $[\hat{a}_1, \hat{a}_1^\dagger] = 1 = [\hat{a}_2, \hat{a}_2^\dagger]$) — note that this Hamiltonian does not conserve the number of particles. Show that you can rewrite the Hamiltonian in the form

$$\hat{H} = \begin{pmatrix} \hat{a}_1^\dagger & \hat{a}_2 \end{pmatrix} \begin{pmatrix} \epsilon & \Delta \\ \Delta & \epsilon \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} - \epsilon \; . \tag{10}$$

7. Find a general transformation to a new set of operators

$$\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2^\dagger \end{pmatrix} = O \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} \tag{11}$$

where O is a 2×2 matrix, such that the bosonic canonical transformation relations are preserved for $\hat{\gamma}_i$.

8. Choose the parameters of this transformation O such that now the Hamiltonian expressed in terms of these new operators is diagonal, i.e., of the form

$$\hat{H} = E_1 \hat{\gamma}_1^\dagger \hat{\gamma}_1 + E_2 \hat{\gamma}_2^\dagger \hat{\gamma}_2 \; . \tag{12}$$