Problem set 3

February 21, 2013

to submit by Monday the 4th of March

1 The weakly-interacting Bose gas

Starting from the Hamiltonian

$$\hat{H} = -\int d\mathbf{r}\hat{\Psi}^\dagger(\mathbf{r})\frac{\hbar^2\nabla^2}{2m}\hat{\Psi}(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r}')U(\mathbf{r}'-\mathbf{r})\hat{\Psi}(\mathbf{r})\hat{\Psi}(\mathbf{r}')q \quad (1)$$

for a system of interacting bosons, where $U(\mathbf{r}'-\mathbf{r}) = \int d\mathbf{q}/(2\pi\hbar)^3 U_\mathbf{q} e^{i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r})/\hbar}$ is the two-body potential,

1. rewrite the Hamiltonian in terms of the creation and annihilation operators, $\hat{a}_\mathbf{p}$ and $\hat{a}^\dagger_\mathbf{p}$, in momentum space $\mathbf{p}$:

$$\hat{\Psi}(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_\mathbf{p} \hat{a}_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} . \quad (2)$$

2. Check that the bosonic commutation relations, $[\hat{a}_\mathbf{p}, \hat{a}_{\mathbf{p}'}] = 0 = [\hat{a}^\dagger_\mathbf{p}, \hat{a}^\dagger_{\mathbf{p}'}]$ and $[\hat{a}_\mathbf{p}, \hat{a}^\dagger_{\mathbf{p}'}] = \delta_{\mathbf{p},\mathbf{p}'}$ are preserved by the Bogoliubov transformation (canonical transformation)

$$\begin{pmatrix} \hat{b}_\mathbf{p} \\ \hat{b}^\dagger_{-\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \cosh\theta_p & -\sinh\theta_p \\ -\sinh\theta_p & \cosh\theta_p \end{pmatrix} \begin{pmatrix} \hat{a}_\mathbf{p} \\ \hat{a}^\dagger_{-\mathbf{p}} \end{pmatrix} . \quad (3)$$

3. Show that instead, for fermionic operators, $\{\hat{c}_{\mathbf{p}\sigma}, \hat{c}_{\mathbf{p}'\sigma'}\} = 0 = \{\hat{c}^\dagger_{\mathbf{p}\sigma}, \hat{c}^\dagger_{\mathbf{p}'\sigma'}\}$ and $\{\hat{c}_{\mathbf{p}\sigma}, \hat{c}^\dagger_{\mathbf{p}'\sigma'}\} = \delta_{\mathbf{p},\mathbf{p}'}\delta_{\sigma,\sigma'}$ (where $\sigma =\uparrow,\downarrow$), the anti-commutation relations are preserved by the following canonical transformation:

$$\begin{pmatrix} \hat{\gamma}_{\mathbf{p}\uparrow} \\ \hat{\gamma}^\dagger_{-\mathbf{p}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_p & \sin\theta_p \\ \sin\theta_p & -\cos\theta_p \end{pmatrix} \begin{pmatrix} \hat{c}_{\mathbf{p}\uparrow} \\ \hat{c}^\dagger_{-\mathbf{p}\downarrow} \end{pmatrix} . \quad (4)$$

4. Show that the canonical transformation (3) diagonalises the reduced Hamiltonian:

$$\hat{H}_{red} = \frac{1}{2}\sum_{\mathbf{p}\neq 0} \begin{pmatrix} \hat{a}^\dagger_\mathbf{p} & \hat{a}_{-\mathbf{p}} \end{pmatrix} \begin{pmatrix} \frac{p^2}{2m} + U_0 n & U_0 n \\ U_0 n & \frac{p^2}{2m} + U_0 n \end{pmatrix} \begin{pmatrix} \hat{a}_\mathbf{p} \\ \hat{a}^\dagger_{-\mathbf{p}} \end{pmatrix} , \quad (5)$$

find the expression for $\cosh\theta_p$, $\sinh\theta_p$, and for the quasi-particle energy $E_p = \sqrt{\frac{U_0 n p^2}{m} + (\frac{p^2}{2m})^2}$.

2 Landau criterion

According to the Landau criterion for superfluidity, quasi-particles can be excited in a fluid where a small defect is moving at a constant velocity $\mathbf{v} = (v, 0)$ (let's consider the 2D case here) if the condition

$$E'_{\mathbf{p}} \equiv E_p - \mathbf{p} \cdot \mathbf{v} = \sqrt{\frac{U_0 n p^2}{m} + (\frac{p^2}{2m})^2} - \mathbf{p} \cdot \mathbf{v} < 0 \tag{6}$$

is satisfied.

5. Show that the defect critical velocity for quasi-particles excitation with a Bogoliubov dispersion $E_p = \sqrt{\frac{U_0 n p^2}{m} + (\frac{p^2}{2m})^2}$ is given by the speed of sound
$$v_c = \min_{\mathbf{p}} \frac{E_p}{p} = c_s = \sqrt{U_0 n/m} \; . \tag{7}$$

6. In 2D ($p^2 = p_x^2 + p_y^2$) find the closed curve Γ in the (p_x, p_y)-plane for which $E'_{\mathbf{p}} = 0$ is satisfied and plot it — N.B. the curve reduces to a point if $v \leq v_c = c_s$.

7. The sound waves propagate from the defect into the fluid when $v > c_s$ with a group velocity $\mathbf{v}_g = \nabla_{\mathbf{p}} E'_{\mathbf{p}}$, the direction of $\mathbf{v}_g$ corresponding to the outward normal direction to the curve Γ found above. Show that for $v > c_s$ the curve Γ has a singularity at $\mathbf{p} = 0$ with a jump in the normal direction given by an angle 2θ such that $sin\theta = c_s/v$ (see Fig. 1).

8. Explain qualitatively the origin and shape of the Cherenkov-like waves observed in real space for supersonic velocities shown in the bottom panels of Fig. 1: For example, why is the period of the waves (in the x-direction of flow) becoming shorter for larger values of v/c_s?

9. Find the critical velocity v_c for a system which spectrum of excitations is gapped, i.e. described by $E_{\mathbf{p}} = \Delta + \frac{p^2}{2m}$ with $\Delta > 0$.

10. Repeat the same exercise now for another gapped spectrum described by $E_{\mathbf{p}} = \sqrt{(\epsilon_{\mathbf{p}} - \delta)(\epsilon_{\mathbf{p}} - \delta + 2U_0 n)}$, where $\delta < 0$. Show that now the critical velocity is always larger than the speed of sound if the "detuning" δ is finite (and negative), and that the limit $v_c = c_s$ is recovered when $\delta \to 0$ — Answer: $v_c = c_s \sqrt{1 - \delta' + \sqrt{-\delta'(-\delta' + 2)}}$, where $\delta' = \delta/U_0 n$.

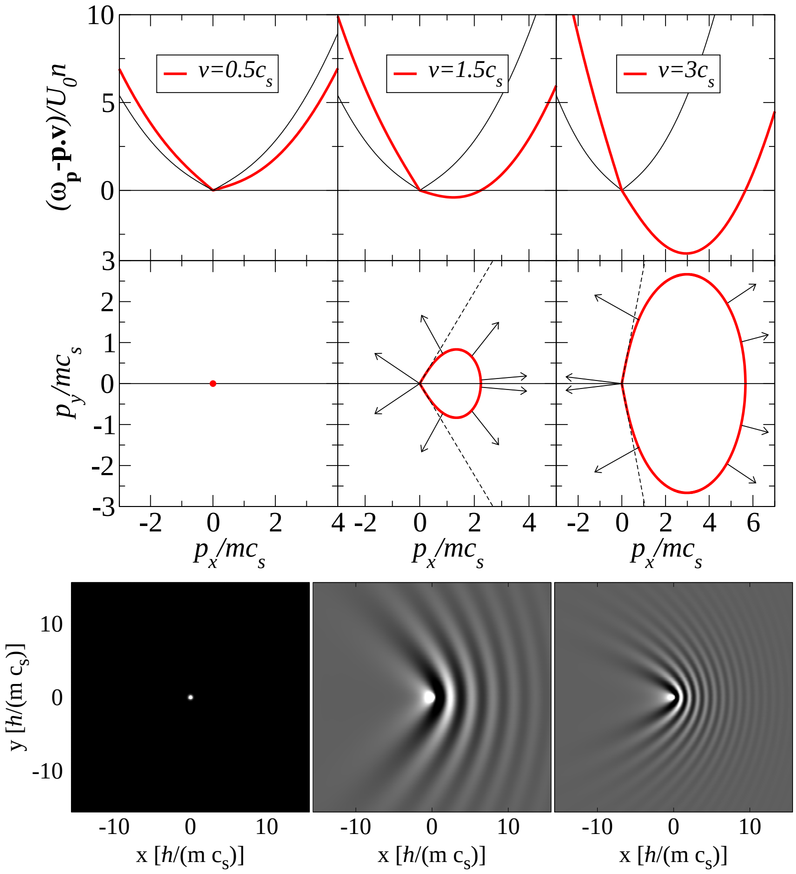

Figure 1: Top row: Tilted spectrum of excitation (i.e., in the reference frame K' of the moving defect), in dimensionless units, $E_{\mathbf{p}}/U_0 n = \sqrt{p'^2/2(p'^2/2+2)} - \mathbf{v}' \cdot \mathbf{p}'$, where $p' = p/mc_s$, $v' = v/c_s$, and where we have assumed that the defect moves in the positive x-direction, $\hat{\mathbf{v}} = (1,0)$. The defect motion is subsonic $v/c_s = 0.5$ in the left panel and supersonic $v/c_s = 1.5$ in the middle and $v/c_s = 3$ in the right panel. Middle row: "Rayleigh" curve Γ in the $\mathbf{p}$-plane such that $E_{\mathbf{p}} - \mathbf{p} \cdot \mathbf{v} = 0$; the arrows in the middle and right panel indicate the direction of emission of phonons, indicating the existence of a Mach cone because of the singularity at the origin of the curve Γ. Bottom row: Condensate density profiles around the defect in the three cases.