Problem set 4

March 5, 2013

to submit by Tuesday the 19^{th} of March

1 Depletion of a Bose-Einstein condensate due to interactions

By making use of the Bogoliubov transformation between the particle operators $a_{\mathbf{p}}$ and the quasi-particle operators $b_{\mathbf{p}}$,

$$\begin{pmatrix} b_{\mathbf{p}} \\ b^{\dagger}_{-\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \cosh\theta_p & -\sinh\theta_p \\ -\sinh\theta_p & \cosh\theta_p \end{pmatrix} \begin{pmatrix} a_{\mathbf{p}} \\ a^{\dagger}_{-\mathbf{p}} \end{pmatrix} , \tag{1}$$

and considering the expression of the quasi-particle energy $E_p = \sqrt{\epsilon_{\mathbf{p}}(\epsilon_{\mathbf{p}} + 2U_0 n)}$,

1. evaluate the number of atoms in the condensate

$$N_0 = N - \sum_{\mathbf{p}\neq 0} f_{\mathbf{p}} , \tag{2}$$

where $f_{\mathbf{p}} = \langle \hat{a}^{\dagger}_{\mathbf{p}} \hat{a}_{\mathbf{p}} \rangle$ is the particle occupation number.

Note that while the particles $\hat{a}_{\mathbf{p}}$ do interact, the quasi-particles $\hat{b}_{\mathbf{p}}$ are non-interacting — within the Bogoliubov expansion we are considering.

2 The Gross-Pitaevskii equation

At zero temperature, $T = 0$, and for a large number of particles N, we can neglect (thermal and) quantum fluctuations $\hat{\psi} \mapsto \psi$ and rewrite the equation of motion in the Heisenberg representation,

$$i\hbar\partial_t \hat{\psi}(\mathbf{r},t) = [\hat{\psi}(\mathbf{r},t), \hat{H}] \tag{3}$$

$$\hat{H} = \int d\mathbf{r} \hat{\Psi}^{\dagger}(\mathbf{r}) \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) \right] \hat{\Psi}(\mathbf{r}) + \frac{U_0}{2} \int d\mathbf{r} \hat{\Psi}^{\dagger}(\mathbf{r}) \hat{\Psi}^{\dagger}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) , \tag{4}$$

as a mean-field equation for the order parameter Ψ (Gross-Pitaevskii equation):

$$i\hbar\partial_t \Psi(\mathbf{r},t) = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + U_0 \left| \Psi(\mathbf{r},t) \right|^2 \right] \Psi(\mathbf{r},t) . \tag{5}$$

2. Rewrite Eq. (5) as two separate equations for the density of the gas, $n(\mathbf{r},t) \equiv |\Psi(\mathbf{r},t)|^2$, and the phase of the order parameter $\phi(\mathbf{r},t)$, where $\psi((\mathbf{r},t)) = \sqrt{n(\mathbf{r},t)}e^{i\phi(\mathbf{r},t)}$.

3. Explain in which limit you can neglect the quantum pressure term $-\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}$.

2.1 Small amplitude oscillations: spectrum of excitations

Let us consider the case where changes in space and time of the order parameter respect to the stationary uniform solution, $\Psi(\mathbf{r},t) = \Psi_0 e^{-i\mu t/\hbar}$ with $\mu = U_0|\Psi_0|^2 = U_0 n$, are small:

$$\Psi(\mathbf{r},t) = [\Psi_0 + \delta\Psi(\mathbf{r},t)]\, e^{-i\mu t/\hbar} \ . \tag{6}$$

4. By substituting (6) in the GPE (5) and linearising in the fluctuation terms $\delta\Psi(\mathbf{r},t)$, re-obtain the Bogoliubov spectrum of excitations, $\hbar\omega_{\mathbf{p}} = \pm\sqrt{\epsilon_{\mathbf{p}}(\epsilon_{\mathbf{p}} + 2U_0 n)}$, by assuming that the small oscillations above the stationary solutions are characterised by one energy $\hbar\omega$ only:

$$\delta\Psi(\mathbf{r},t) = \sum_{\mathbf{p}} \left[u_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{r}-\hbar\omega t)/\hbar} + v^*_{\mathbf{p}} e^{-i(\mathbf{p}\cdot\mathbf{r}-\hbar\omega t)/\hbar}\right] \ . \tag{7}$$

5. Describe what happens if the interaction between the bosonic particles is attractive, i.e. $U_0 < 0$. For example evaluate the compressibility of the interacting gas and the Bogoliubov spectrum and analyse the results you get.

2.2 Condensates in hard wall potentials

6. Show that the condensate wavefunction $\Psi(\mathbf{r})$ in a hard wall potential can be written, near the wall, as

$$\Psi(x) = \sqrt{\bar{n}}\tanh(\frac{x}{\sqrt{2}\xi}) \ , \tag{8}$$

where x is the distance from the wall, $\bar{n}$ is the density of the condensate very far from the wall, and ξ is the healing length, $\xi = \hbar/\sqrt{2mU_0\bar{n}}$.