Problem set 5

March 5, 2013

to submit by Thursday the 21st of March

0.1 Second and first order phase transitions

Let us consider the following phenomenological free energy density of a given system described in terms of a real homogeneous order parameter m (Landau's mean-field theory):

$$\frac{f(m)}{V} = f_0 + a(T - T_c)m^2 + \frac{b}{2}m^4 \ , \tag{1}$$

where $a > 0$ and $b > 0$ are two constant positive parameters.

1. If the state that minimises the free energy is the physically realised state, describe the second order phase transition across $T = T_c$ and find how the minimum m_0 changes with the temperature.

2. Now let's immagine that a system is instead described by the following free energy

$$\frac{f(m)}{V} = f_0 + a(T - T_c)m^2 + \frac{b}{2}m^4 + \frac{c}{3}m^6 \ , \tag{2}$$

   where the coefficient $c > 0$ for stability, but the coefficient of the fourth order term, b, can now change sign (as before $a > 0$). Show that, for $T > T_c$, two (symmetric) secondary minima develop in the free energy density at finite values of the order parameter m and that, for a critical temperature $T = T_{first} > T_c$, a first order transition can be indentified where the minimum for $m = 0$ and the one for $m \neq 0$ have the same energy. Describe the system phase diagram as a function of the system parameters.

1 Two-component condensates

Let us consider the case of a gas formed by a mixture of two types of bosons of equal densities. We indicate with $\psi_\uparrow$ and $\psi_\downarrow$ the BEC order parameter for each boson — we can think of them as, e.g., the same boson in two different spin state. Now the system is characterised by an intra-species interaction strength $U_{\uparrow\uparrow} = U_{\downarrow\downarrow} = U_0$ (which we are assuming to be the same for the two species and repulsive) and an inter-species interaction strength, which we indicate as

$U_{\uparrow\downarrow} = U_0 - 2U_1$, and that can be either attractive or repulsive. The free energy of the system can be thus written as:

$$\begin{aligned} E[\psi_\uparrow, \psi_\uparrow^*, \psi_\downarrow, \psi_\downarrow^*] = \int d\mathbf{r} \Bigg[& \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^* \left(-\frac{\hbar^2 \nabla^2}{2m_\sigma} - \mu \right) \psi_\sigma \\ & + \frac{U_0}{2} \left(|\psi_\uparrow|^4 + |\psi_\downarrow|^4 \right) + (U_0 - 2U_1) |\psi_\uparrow|^2 |\psi_\downarrow|^2 \Bigg] \\ = \int d\mathbf{r} \Bigg[& \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^* \left(-\frac{\hbar^2 \nabla^2}{2m_\sigma} - \mu \right) \psi_\sigma \\ & + \frac{U_0 - U_1}{2} \left(|\psi_\uparrow|^2 + |\psi_\downarrow|^2 \right)^2 + \frac{U_1}{2} \left(|\psi_\uparrow|^2 - |\psi_\downarrow|^2 \right)^2 \Bigg] \ . \end{aligned} \tag{3}$$

3. Evaluate the two coupled Gross-Pitaevskii equations associated to (3).

4. Consider now the case of uniform field solutions. Show that the the free energy (3) has a minimum for $|\psi_\uparrow| = |\psi_\downarrow|$ if $U_1 > 0$, while the minimum occurs for either $(|\psi_\uparrow| \neq 0, |\psi_\downarrow| = 0)$ or $(|\psi_\uparrow| = 0, |\psi_\downarrow| \neq 0)$ when $U_1 < 0$ — Note that, for stability, one has to require that $U_0 > 0$ and $U_0 - U_1 > 0$. Discuss what happens when either $U_0 < 0$ or $U_0 - U_1 < 0$. Find the values of the minima in terms of the chemical potential. Plot the system phase diagram in the (U_0, U_1) space.

5. Evaluate the Bogoliubov spectra of excitation separately for both cases $U_1 > 0$ and $U_1 < 0$ and discuss your results — now assume that $U_0 > 0$ and $U_0 - U_1 > 0$.

2 Vortex line solutions

We have derived in class that a solution of the time-independent Gross-Pitaevskii equation (GPE) is the one describing a vortex line of charge j:

$$\Psi_v(\mathbf{r}) = e^{ij\varphi} |\Psi_v(r)| \ , \tag{4}$$

where (r, φ, z) are the cilindrical coordinates, i.e. φ is the azymuthal angle.

6. Show that, by parametrising the amplitude of the vortex solution $|\Psi_v(r)|$ in terms of a dimensionless function of $\tilde{r} = r/\xi$, where $\xi = \hbar/\sqrt{2mU_0\bar{n}}$ is the healing length and $\bar{n} \simeq \mu/U_0$ is the uniform density solution,

$$|\Psi_v(r)| = \sqrt{\bar{n}} f(\tilde{r}) \ , \tag{5}$$

the GPE can be equivalently rewritten as

$$\frac{1}{\tilde{r}} \frac{d}{d\tilde{r}} \left(\tilde{r} \frac{df(\tilde{r})}{d\tilde{r}} \right) + \left(1 - \frac{j^2}{\tilde{r}^2} \right) f(\tilde{r}) - f^3(\tilde{r}) = 0 \ . \tag{6}$$

7. Show that (in a cilindrical container of height L and radius R, thus with volume $V = L\pi R^2$) the free energy of the vortex solution minus the free

energy of the homogeneous solution ($E'_0 = V(\frac{U_0\bar{n}^2}{2} - \mu\bar{n}) = -\frac{L\pi R^2}{2}U_0\bar{n}^2$) can be written as

$$E'_v - E'_0 = \frac{\pi L n\hbar^2}{m}\int_0^{R/\xi} d\tilde{r}\tilde{r}\left[\left(\frac{df}{d\tilde{r}}\right)^2 + \frac{j^2}{\tilde{r}^2}f^2 + \frac{1}{2}\left(f^2-1\right)^2\right] . \qquad (7)$$

Note that we use the notation $E' = E - \mu N$.

8. For the case of a vortex of charge $|j| = 1$, consider the following variational solution

$$f(\tilde{r}) = \frac{\tilde{r}}{\sqrt{\alpha + \tilde{r}^2}} , \qquad (8)$$

and use it as a trial form for the real solution $f(\tilde{r})$ of (6), i.e., substitute (8) into (7) and minimise the energy expression with respect to the parameter in the trial function α. Show that the optimal value is $\alpha = 2$.

9. (**Optional:**) Show that for the optimal solution, one gets

$$E'_v - E'_0 = \frac{\pi L n\hbar^2}{m}\ln\left(\frac{1.497R}{\xi}\right) , \qquad (9)$$

which is very close to the exact (numerical) result $E'_v - E'_0 = \frac{\pi L n\hbar^2}{m}\ln(1.464R/\xi)$.