Problem set 6

March 21, 2013

to submit by Thursday the 4^{th} of April

1 Thomas-Fermi limit

In the Thomas-Fermi approximation, the density profile of the interacting gas is given by

$$n^{\mathrm{TF}}(\mathbf{r}) = \frac{\mu - V_{\mathrm{ext}}(\mathbf{r})}{U_0} , \tag{1}$$

where the chemical potential μ can be found from the normalisation condition $\int d\mathbf{r} n^{\mathrm{TF}}(\mathbf{r}) = N$.

1. In the case of an isotropic harmonic trapping potential, $V_{\mathrm{ext}}(\mathbf{r}) = \frac{1}{2}m\bar{\omega}^2 r^2$, evaluate the expression of the chemical potential μ as a function of the number of particles in the gas, N, the trap frequency $\bar{\omega}$, the harmonic oscillator length $a_{\bar{\omega}} = \sqrt{\hbar/(m\bar{\omega})}$, and the s-wave scattering length $a = mU_0/(4\pi\hbar^2)$.

2. For which values of the system parameters do you expect the Thomas-Fermi approximation to be valid?

2 Times of flight measurements

3. Demonstrate that the free propagation of a wave-packet $\chi_0(\mathbf{r}, t)$ can be described analytically by:

$$\chi_0(\mathbf{r}, t) = \left(\frac{m}{2\pi\hbar t}\right)^{3/2} \int d\mathbf{r}' \chi_0(\mathbf{r}', 0) e^{i\frac{m}{2\hbar t}(\mathbf{r}-\mathbf{r}')^2} . \tag{2}$$

(Hint: show that $\chi_0(\mathbf{r}, t)$ satisfies $i\hbar\partial_t\chi_0 = -\frac{\hbar^2\nabla^2}{2m}\chi_0$).

4. Evaluate the expansion of a Gaussian wave-packet

$$\chi_0(\mathbf{r}, 0) = \left(\frac{m\bar{\omega}}{\pi\hbar}\right)^{3/4} e^{-\frac{r^2}{2a_{\bar{\omega}}^2}} \tag{3}$$

from a trap, i.e. evaluate $\chi_0(\mathbf{r}, t)$ according to Eq. (2). Describe how amplitude $|\chi_0(\mathbf{r}, t)|$ and phase $\phi_0(\mathbf{r}, t)$ change in time, where $\chi_0(\mathbf{r}, t) = |\chi_0(\mathbf{r}, t)|e^{i\phi_0(\mathbf{r},t)}$. Evaluate the velocity field $\mathbf{v}_s = \frac{\hbar}{m}\nabla\phi_0$ and check that for long times, $\bar{\omega}t \gg 1$, follows the classical law $\mathbf{v}_s \simeq \mathbf{r}/t$.

5. Show that you can write

$$|\chi_0(\mathbf{r},t)| \underset{t\bar{\omega}\gg 1}{\simeq} \chi_0(\mathbf{p},0) \ , \tag{4}$$

where $\chi_0(\mathbf{p},0)$ is the Fourier transform in momentum space of the Gaussian wave-packet (3) and $\mathbf{p} = m\mathbf{r}/t$.

6. Demonstrate that the thermal component of a Bose gas, $n_T(\mathbf{r},t)$, which in the semi-classical approximation can be written as

$$n_T(\mathbf{r},t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} f_{\mathbf{p}}(\mathbf{r} - \frac{\mathbf{p}}{m}t) \ , \tag{5}$$

where

$$f_{\mathbf{p}}(\mathbf{r}) = \frac{1}{e^{\beta[\frac{p^2}{2m} + V_{\text{ext}}(\mathbf{r}) - \mu]} - 1} \ , \tag{6}$$

and $V_{\text{ext}}(\mathbf{r}) = \frac{1}{2}m\bar{\omega}^2 r^2$, for long times $t\bar{\omega} \gg 1$ coincides with the thermal distribution in momentum space:

$$\lim_{t\bar{\omega}\gg 1} n_T(\mathbf{r},t) = \frac{m^3}{t^3} n_T(\mathbf{p}) \ . \tag{7}$$

In other words, when the gas is released from the trap, its thermal component expands and the space distributions at long times coincides with the momentum one.

7. Why can we say that the thermal cloud expands isotropically, even if initially the trap might be not isotropic, say for $V_{\text{ext}}(\mathbf{r}) = \frac{1}{2}m(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2)$?

3 Interference between two condensates

Consider the result from the exercise n. 4. above, where you have evaluated the free time evolution of the Gaussian wave-packet and know the explicit expression of its phase $\phi_0(\mathbf{r},t)$.

8. Describe the interference between two condensates, one centered at the point $-\mathbf{d}/2$ and the other at $\mathbf{d}/2$. Plot the shape of the interference fringe pattern in this geometry and derive the expression of the fringe period in space at a given time t after the expansion.

9. What happens to the pattern of the interference fringes if one of the condensates contains a vortex of charge $j = 1$? (Hint: in the worse case scenario, ask Carlos Antón for help!).