Problem set 7

April 5, 2013

to submit by Monday the 15th of April

1 The homogeneous ideal Fermi gas in 3D and in 2D

Consider first a homogeneous gas of N free fermions in a single spin state (no spin degeneracy) in a 3D volume V.

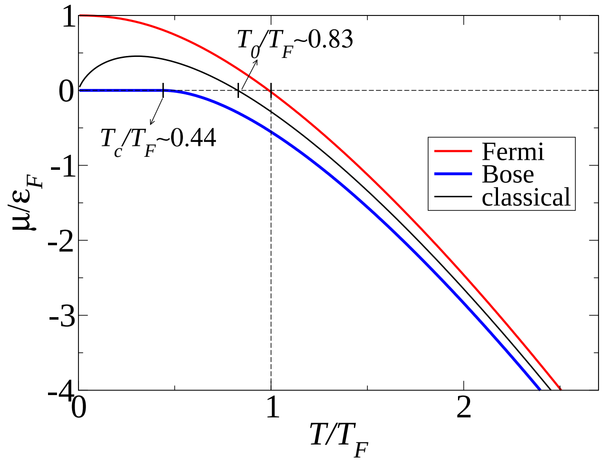
1. Evaluate the Fermi energy

$$\varepsilon_F = \frac{\hbar^2}{2m}(6\pi^2 n)^{2/3} \tag{1}$$

as a function of the gas density, $n = N/V$, and the particle mass m.

2. Show that the ground state energy of the system at zero temperature is given by $E(T=0) = \frac{3}{5}N\varepsilon_F$ and evaluate the gas compressibility $\kappa^{-1} = -V\frac{\partial P}{\partial V}$, where the pressure is related to the internal energy by $PV = \frac{2}{3}E$. Compare and comment the result you get with the one you got for an ideal gas of bosons.

3. Evaluate numerically the expression of the chemical potential $\mu(T)/k_B T_F$ as a function of the temperature T/T_F, where the Fermi temperature is given by $T_F = k_B\varepsilon_F$. Compare the plot you get with the case of a classical gas of single component particles (here you can do the calculation analytically), where you define the temperature $k_B T_0 = \frac{2\pi\hbar^2}{m}n^{2/3}$ as $n\lambda_{T_0}^3 = 1$, and also with the case of a gas of identical bosons, where the critical temperature for BEC is given by $k_B T_c = \frac{2\pi\hbar^2}{m}n^{2/3}/[g_{3/2}(1)]^{2/3}$ (you got the plot in the problem set 1). Your result should look like the one shown in the figure here below. Comment the results you get. Why the critical temperature for BEC is the same order of magnitude of the Fermi energy, $T_c \sim T_F \sim T_0$?



4. Show that for fermions, the following low temperature, $T \ll T_F$, expansions apply for the chemical potential and the system internal energy respectively:

$$\mu(T) \simeq \varepsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \dots\right]$$
$$E(T) \simeq \frac{3}{5} N\varepsilon_F \left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] .$$

5. **Optional:** Plot the system internal energy $E(T)$ and heat capacity $c_v = \frac{\partial E(T)}{\partial T}$ as a function of the temperature T/T_F for fermions, bosons and a classical gas.

6. Consider now the case of ^{3}He atoms (fermions with spin $S = 1/2$). If near absolute zero the density times the mass is $mn = 0.081$ g cm^{-3}, evaluate the Fermi energy and the Fermi temperature.

7. Consider now a homogeneous gas of identical fermions (single spin state) in 2D: evaluate the Fermi energy and, at finite temperature, establish that:

$$\varepsilon_F = \mu + k_B T \log\left(1 + e^{-\beta\mu}\right) . \tag{2}$$

Note that in 2D the density of state is constant, and the integral fixing the number of particles can be evaluated analytically. Invert this expression and plot $\mu/k_B T_F$ versus T/T_F together with the result you get for the chemical potential of a classical gas in 2D. What happens for a homogeneous gas of bosons in 2D?

2 The ideal Fermi gas in a 3D harmonic trap

Consider a gas of N identical fermions (no spin) in a three-dimensional harmonic trap, $V_{ext}(\mathbf{r}) = \frac{1}{2} m\bar{\omega}^2 r^2$.

8. Evaluate the Fermi energy, ε_F, as a function of the trap frequency $\bar{\omega}$ and the number of particles N.

9. Compare the expression you get for the Fermi temperature, $k_B T_F = \varepsilon_F$, with the one you got for the transition temperature of N identical bosons in a 3D trap to a BEC state, $k_B T_c \simeq 0.94\hbar\bar{\omega}N^{1/3}$. What can you conclude?

10. Evaluate the gas internal energy $E(T)$ and plot the energy per particle in units of the Fermi energy, $E(T)/(N\epsilon_F)$ as a function of T/T_F. Show that at zero temperature, $E(T=0) = \frac{3}{4}N\varepsilon_F$. Derive explicitly the low temperature behaviour, $T \ll T_F$ and show that, at high temperatures, you recover the classical result $E_{cl}(T) = 3Nk_B T$ (note that in a trap you have 3 rather than $\frac{3}{2}$).

Let us assume that, in the limit of a large number of fermions N, one can make use of a semiclassical description, where the properties of the gas at a given point $\mathbf{r}$ of the trap are assumed to be those of a uniform gas having a density equal to the local density $n(\mathbf{r})$. In this *local density approximation* the Fermi distribution function and the local density $n(\mathbf{r})$ are given respectively by

$$f^F(\mathbf{r},\mathbf{p}) = \frac{1}{\exp[\beta(\epsilon_{\mathbf{p}} + V_{ext}(\mathbf{r}) - \mu)] + 1} \tag{3}$$

$$n(\mathbf{r}) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} f^F(\mathbf{r},\mathbf{p}) \ , \tag{4}$$

where $\epsilon_{\mathbf{p}} = p^2/2m$.

11. Re-obtain the expression of the Fermi energy for a trapped gas you got previously by requiring that $N = \int d\mathbf{r} n(\mathbf{r})$.

12. Derive the density profile $n(\mathbf{r})$ in the zero temperature limit.