Problem set 8

April 16, 2013

to submit by Thursday the 25^{th} of April

1 The one-pair Cooper problem

Consider a pair of electrons with opposite spins and positions $\mathbf{r}_1$ and $\mathbf{r}_2$, on top of a Fermi sphere of radius k_F, $k > k_F$, described by the following pair wavefunction:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \varphi(\mathbf{r}) \frac{|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}} \qquad \varphi(\mathbf{r}) = \frac{1}{V} \sum_{\mathbf{k}} \varphi_{\mathbf{k}} e^{i\mathbf{r}\cdot\mathbf{k}} , \tag{1}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and where we have assumed the center of mass wavevector $\mathbf{q}$ to be zero. The Schrödinger equation for the pair relative wavefunction can be written in momentum space as

$$2\xi_{\mathbf{k}}\varphi_{\mathbf{k}} + \frac{1}{V}\sum_{\mathbf{k}'} U_{\mathbf{k}-\mathbf{k}'}\varphi_{\mathbf{k}'} = E\varphi_{\mathbf{k}} , \tag{2}$$

where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \varepsilon_F$. We approximate the interaction with

$$U_{\mathbf{k}-\mathbf{k}'} = \begin{cases} U_0 = -|U_0| & |\xi_{\mathbf{k}}|, |\xi_{\mathbf{k}'}| < \hbar\omega_D \\ 0 & \text{otherwise} . \end{cases} \tag{3}$$

1. Show that the normal gas is unstable towards the formation of a pair bound state, $E < 0$, for any value of the attractive interaction strength $U_0 < 0$, and find the expression of E. Give an estimate of E for $N(\varepsilon_F)|U_0| \simeq 0.2$ ($N(\varepsilon_F) = m^{3/2}/(\sqrt{2}\pi^2\hbar^3)$) and $T_D \simeq 300\text{K}$ ($\hbar\omega_D = k_B T_D$).

2. Show that a necessary condition for the above situation is the Pauli blocking — i.e., show that for $\varepsilon_F = 0$, not all values of $U_0 < 0$ imply the existence of a bound state.

3. Estimate the size of the Cooper pair

$$\sqrt{\langle r^2 \rangle} = \left(\frac{\int d\mathbf{r} r^2 |\varphi(\vec{r})|^2}{\int d\mathbf{r} |\varphi(\vec{r})|^2} \right)^{1/2} \simeq \sqrt{\frac{4}{3}\frac{\hbar v_F}{E}} , \tag{4}$$

and give an estimate for $v_F \simeq 10^8$ cm s^{-1}.

4. Explain why the lowest energy Cooper pair is the one obtained from electrons of opposite wavevectors and spins, $(\mathbf{k}, \uparrow)$ and $(-\mathbf{k}, \downarrow)$, i.e., a pair with zero center of mass momentum $\mathbf{q} = 0$. (**Optional**:) Show that

$$E(\mathbf{q}) = E + \frac{\hbar v_F}{2} q \ . \tag{5}$$

2 Coherent states versus Fock states

Consider the following coherent state

$$|\phi\rangle_c = e^{-|\phi|^2/2} e^{\phi \hat{b}^\dagger} |0\rangle \ , \tag{6}$$

where $\phi \in \mathbb{C}$, and $\hat{b}^\dagger$ and $\hat{b}$ are the bosonic creation and annihilation operators, $[\hat{b}, \hat{b}^\dagger] = 1$ and $[\hat{b}^\dagger, \hat{b}^\dagger] = 0 = [\hat{b}, \hat{b}]$.

5. Show that the coherent state $|\phi\rangle_c$ is normalised, ${}_c\langle\phi|\phi\rangle_c = 1$ and that it is an eigenstate of the annihilation operator

$$\hat{b}|\phi\rangle_c = \phi|\phi\rangle_c \ . \tag{7}$$

In addition, evaluate $\hat{b}^\dagger|\phi\rangle_c$.

6. Evaluate the product between two coherent states ${}_c\langle\theta|\phi\rangle_c$.

7. Show that the average number of particles and its variance in a coherent state are give respectively by:

$$\bar{N} = {}_c\langle\phi|\hat{b}^\dagger\hat{b}|\phi\rangle_c = |\phi|^2 \qquad\qquad \frac{\sigma}{\bar{N}} = \frac{1}{\sqrt{\bar{N}}} \ . \tag{8}$$

Now consider the Fock state (for which the number of particles is fixed to N):

$$|\Psi_N\rangle = \frac{1}{\sqrt{N!}} (\hat{b}^\dagger)^N |0\rangle \tag{9}$$

8. Show that the number of particles N and the phase θ are conjugate variables, i.e., that you can write the Fock state $|\Psi_N\rangle$ as a linear superposition of coherent states $|e^{i\theta}\rangle_c$:

$$|\Psi_N\rangle \propto \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta N} |e^{i\theta}\rangle_c \ . \tag{10}$$

Here, the proportionality symbol $\propto$ means we are not going to worry about normalisation terms.

9. Explain in which sense you can interpret (by considering N as a continuous variable):

$$N \leftrightarrow -i\frac{\partial}{\partial\theta} \qquad\qquad \theta \leftrightarrow -i\frac{\partial}{\partial N} \ . \tag{11}$$