Subharmonic Shapiro Steps and Assisted Tunneling in Superconducting Point Contacts

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We analyze the current in a superconducting point contact of arbitrary transmission in the presence of a microwave radiation. The interplay between the ac Josephson current and the microwave signal gives rise to Shapiro steps at voltages $V = (m/n)\hbar\omega_r/2e$, where n, m are integer numbers and ω_r is the radiation frequency. The *subharmonic steps* ($n \neq 1$) are a consequence of multiple Andreev reflections (MAR) and provide a signature of the peculiar ac Josephson effect at high transmission. Moreover, the dc current exhibits a rich subgap structure due to photon-assisted MARs.

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Introduction.— Our understanding of the electronic transport through superconducting nanostructures has experienced a notable development in the last few years [1]. This has partly been due to the appearance of the metallic atomic-size contacts, which can be produced by means of a scanning tunneling microscope and break-junction techniques [2–5]. These nanowires have turned out to be ideal systems to test the modern transport theories in mesoscopic superconductors. Thus, for instance, Scheer and co-workers [3] found a quantitative agreement between the measurements of the current-voltage characteristics of different atomic contacts and the predictions of the theory for a single-channel superconducting contact [6,7]. These experiments not only helped to clarify the structure of the subgap current in superconducting contacts, but they also showed that the set of the transmission coefficients in an atomic-size contact is amenable to measurement. This possibility has recently allowed a set of experiments that confirm the theoretical predictions for transport properties such as supercurrent [4] and noise [5]. From these combined theoretical and experimental efforts a coherent picture of transport in superconducting point contacts has emerged with multiple Andreev reflections (MAR) [8] as a central concept. However, in spite of these recent successes, one of the most remarkable predictions of MAR theory remains to be confirmed, namely, the *ac Josephson effect*. The theory says that in a constant voltage biased superconducting point contact, the time-dependent current is given by $I(t) = \sum_n I_n e^{in\omega_0 t}$. This means that the occurrence of MARs gives rise to the appearance of alternating currents that oscillate not only with the Josephson frequency $\omega_0 = 2eV/\hbar$, V being the voltage, as in the case of tunnel junctions, but also with all its harmonics. The direct observation of these alternating components is prevented by their high oscillation frequencies, and an indirect method is required to probe their existence.

In this Letter, we present a theoretical analysis of the current in a superconducting point contact under a microwave radiation [9]. We show that the interplay between the ac Josephson current components and a microwave signal leads to the appearance of Shapiro steps at voltages $V = (m/n)\hbar\omega_r/2e$, where n, m are integer numbers and ω_r is the frequency of the radiation. This means that in addition to the usual steps ($n = 1$) found in tunnel junctions [10], there also appear *subharmonic Shapiro steps* ($n \neq 1$), which constitute an unambiguous signature of the ac Josephson effect in these contacts. Moreover, we also find that the dc background current, in which the Shapiro steps are superimposed, exhibits a rich subgap structure, which can be understood in terms of photon-assisted MARs and provides a natural explanation of experimental findings in the early 1970s [11].

Theoretical model.— Our goal is to calculate the current in a voltage biased superconducting quantum point contact (SQPC) [12] in the presence of a monochromatic radiation of frequency ω_r. We assume that the external radiation produces an effective time-dependent voltage $V(t) = V + V_{ac}\sin\omega_r t$. Our task is to extend the MAR theory to the case of such a time-dependent voltage, for which the so-called Hamiltonian approach [7] is a convenient starting point. For the voltage range $eV \sim \Delta$ one can neglect the energy dependence of the transmission coefficients and all transport properties can be expressed as a superposition of independent channel contributions. Thus, the problem reduces to the analysis of a single channel contact, which can be described by means of the following tight-binding-like Hamiltonian [7]:

$$\hat{H} = \hat{H}_L + \hat{H}_R + \sum_\sigma \{v c^\dagger_{L\sigma} c_{R\sigma} + v^* c^\dagger_{R\sigma} c_{L\sigma}\}, \quad (1)$$

where $H_{L,R}$ are the BCS Hamiltonians for the isolated electrodes. In the coupling term L and R stand for the outermost sites of each electrode, and v is a hopping parameter coupling these sites. This parameter determines the normal transmission coefficient of this model $\mathcal{T}$, which adopts the form $\mathcal{T} = 4(v/W)^2/[1 + (v/W)^2]^2$, where $W = 1/\pi\rho_F$, with ρ_F being the electrodes density of states at the Fermi energy [7].

In this model the current evaluated at the interface between the two electrodes adopts the form

$$I(t) = \frac{ie}{\hbar}\sum_\sigma \{v\langle c^\dagger_{L\sigma}(t)c_{R\sigma}(t)\rangle - v^*\langle c^\dagger_{R\sigma}(t)c_{L\sigma}(t)\rangle\}. \quad (2)$$

The nonequilibrium expectation values in Eq. (2) can be expressed in terms of the Keldysh Green functions $\hat{G}^{+,-}_{i,j}$, which in the 2×2 Nambu representation read

$$\hat{G}^{+-}_{i,j}(t,t') = i\begin{pmatrix} \langle c^\dagger_{j\uparrow}(t')c_{i\uparrow}(t)\rangle & \langle c_{j\downarrow}(t')c_{i\uparrow}(t)\rangle \\ \langle c^\dagger_{j\uparrow}(t')c^\dagger_{i\downarrow}(t)\rangle & \langle c_{j\downarrow}(t')c^\dagger_{i\downarrow}(t)\rangle \end{pmatrix}. \quad (3)$$

Thus, the current can now be written as

$$I(t) = \frac{e}{\hbar}\mathrm{Tr}\{\hat{\tau}_3[\hat{v}(t)\hat{G}^{+-}_{RL}(t,t) - \hat{v}^\dagger(t)\hat{G}^{+-}_{LR}(t,t)]\}, \quad (4)$$

where $\hat{\tau}_3$ is the corresponding Pauli matrix, Tr denotes the trace in Nambu space, and $\hat{v}$ is the hopping that in the Nambu matrix representation is written as

$$\hat{v}(t) = \begin{pmatrix} ve^{i\phi(t)/2} & 0 \\ 0 & -v^* e^{-i\phi(t)/2}\end{pmatrix}. \quad (5)$$

Here, $\phi(t) = \phi_0 + \omega_0 t + 2\alpha\cos\omega_r t$ is the time-dependent superconducting phase difference. The constant $\alpha = eV_{ac}/(\hbar\omega_r)$ measures the strength of the coupling to the electromagnetic field and is proportional to the square root of the radiation power.

In order to determine the Green functions we follow a perturbative scheme and treat the coupling term in Hamiltonian (1) as a perturbation. The unperturbed Green functions, $\hat{g}$, correspond to the uncoupled electrodes in equilibrium. Thus, the retarded and advanced components adopt the BCS form: $\hat{g}^{r,a}(\epsilon) = g^{r,a}(\epsilon)\hat{1} + f^{r,a}(\epsilon)\hat{\tau}_1$, where $g^{r,a}(\epsilon) = -(\epsilon^{r,a}/\Delta)f(\epsilon) = -\epsilon^{r,a}/W\sqrt{\Delta^2 - (\epsilon^{r,a})^2}$, where $\epsilon^{r,a} = \epsilon \pm i\eta$, with $\eta = 0^+$. One can express the current in a more compact way in terms of the T matrix. The T matrix associated to the time-dependent perturbation of Eq. (5) is defined as $\hat{T}^{r,a} = \hat{v} + \hat{v}\circ\hat{g}^{r,a}\circ\hat{T}^{r,a}$, where the $\circ$ product is a shorthand for integration over intermediate time arguments. As shown in Ref. [7], the current in terms of the T-matrix components reads

$$I(t) = \frac{e}{\hbar}\mathrm{Tr}[\hat{\tau}_3(\hat{T}^r_{LR}\circ\hat{g}^{+-}_R\circ\hat{T}^a_{RL}\circ\hat{g}^a_L - \hat{g}^r_L\circ\hat{T}^r_{LR}\circ\hat{g}^{+-}_R\circ\hat{T}^a_{RL} + \hat{g}^r_R\circ\hat{T}^r_{RL}\circ\hat{g}^{+-}_L\circ\hat{T}^a_{LR} - \hat{T}^r_{RL}\circ\hat{g}^{+-}_L\circ\hat{T}^a_{LR}\circ\hat{g}^a_R)]. \quad (6)$$

In order to solve the T-matrix integral equation it is convenient to Fourier transform with respect to the temporal arguments, $\hat{T}(t,t') = (1/2\pi)\int d\epsilon\int d\epsilon' e^{-i\epsilon t}e^{i\epsilon' t'}\hat{T}(\epsilon,\epsilon')$. Because of time dependence of the coupling element [see Eq. (5)], one can show that $\hat{T}(\epsilon,\epsilon')$ admits the following solution: $\hat{T}(\epsilon,\epsilon') = \sum_{n,m}\hat{T}(\epsilon,\epsilon + neV + m\hbar\omega_r)\delta(\epsilon - \epsilon' + neV + m\hbar\omega_r)$. Thus, one can finally write down the current as $I(t) = \sum_{n,m} I^m_n \exp[i(n\phi_0 + n\omega_0 t + m\omega_r t)]$, where the current amplitudes I^m_n can be expressed in terms of the T-matrix Fourier components, $\hat{T}^{kl}_{nm} \equiv \hat{T}(\epsilon + neV + k\hbar\omega_r, \epsilon + meV + l\hbar\omega_r)$, in the following way:

$$I^m_n = \frac{e}{h}\int d\epsilon\sum_{i,k}\mathrm{Tr}[\hat{\tau}_3(\overset{r}{\hat{T}}{}^{0k}_{LR,0i}\overset{+-}{\hat{g}}{}^k_{R,i}\overset{a}{\hat{T}}{}^{km}_{RL,in}\overset{a}{\hat{g}}{}^m_{L,n} - \overset{r}{\hat{g}}{}^0_{L,0}\overset{r}{\hat{T}}{}^{0k}_{LR,0i}\overset{+-}{\hat{g}}{}^k_{R,i}\overset{a}{\hat{T}}{}^{km}_{RL,in} + \overset{r}{\hat{g}}{}^0_{R,0}\overset{r}{\hat{T}}{}^{0k}_{RL,0i}\overset{+-}{\hat{g}}{}^k_{L,i}\overset{a}{\hat{T}}{}^{km}_{LR,in} - \overset{r}{\hat{T}}{}^{0k}_{RL,0i}\overset{+-}{\hat{g}}{}^k_{L,i}\overset{a}{\hat{T}}{}^{km}_{LR,in}\overset{a}{\hat{g}}{}^m_{R,n})]. \quad (7)$$

At this point, the calculation of the current has been reduced to determination of the Fourier components of the T matrix. In the case of a symmetric contact considered here, one can show that the dc current can be expressed only in terms of $\hat{T}^k_i \equiv \overset{a}{\hat{T}}{}^{k0}_{LR,i0}$, which fulfill the following set of linear algebraic equations:

$$\hat{T}^k_i = \hat{v}^k_i + \sum_l\{\hat{\mathcal{E}}^{kl}_{i,i}\hat{T}^l_i + \hat{\mathcal{V}}^{kl}_{i,i+2}\hat{T}^l_{i+2} + \hat{\mathcal{V}}^{kl}_{i,i-2}\hat{T}^l_{i-2}\}, \quad (8)$$

where the different matrix coefficients adopt the following form in terms of the unperturbed Green functions:

$$\hat{v}^k_i = \frac{v}{2}J_k(\alpha_0)[i^k(\hat{1} + \hat{\tau}_3)\delta_{i,-1} - (-i)^k(\hat{1} - \hat{\tau}_3)\delta_{i,1}],$$

$$\hat{\mathcal{E}}^{kl}_{i,i} = v^2 i^{k+l}\sum_j(-1)^j J_{k-j}(\alpha)J_{j-l}(\alpha)\begin{pmatrix} g^j_{i+1}g^l_i & g^j_{i+1}f^l_i \\ g^j_{i-1}f^l_i & g^j_{i-1}g^l_i\end{pmatrix},$$

$$\hat{\mathcal{V}}^{kl}_{i,i+2} = -v^2 i^{k-l}\sum_j J_{k-j}(\alpha)J_{j-l}(\alpha)f^j_{i+1}\begin{pmatrix} f^l_{i+2} & g^l_{i+2} \\ 0 & 0\end{pmatrix},$$

$$\hat{\mathcal{V}}^{kl}_{i,i-2} = -v^2 i^{l-k}\sum_j J_{k-j}(\alpha)J_{j-l}(\alpha)f^j_{i-1}\begin{pmatrix} 0 & 0 \\ g^l_{i-2} & f^l_{i-2}\end{pmatrix},$$

where we have used the shorthand notation $\hat{g}^k_i = \hat{g}^a(\epsilon + ieV + k\hbar\omega_r)$ and $J_n(\alpha)$ is the Bessel function of order n. In some limits one can find an analytical solution of these systems, but in general a numerical calculation is needed.

Results and discussions.—Let us concentrate on the dc current, I_{dc}. This current is the sum of two contributions: $I_{\mathrm{dc}} = I_B + I_{\mathrm{Shapiro}}$, where $I_B \equiv I^0_0$ is a background current and $I_{\mathrm{Shapiro}} = \sum_{n,m} I^m_n e^{in\phi_0}\delta(V - V^m_n)$ is the Shapiro steps contribution at discrete voltages $V_n^m = (m/n)\hbar\omega_r/2e$. Notice that several ac current amplitudes can give a dc contribution at the same voltage. Notice also that the Shapiro step contribution depends on the average value of the phase, ϕ_0. We concentrate on the height of the Shapiro steps, which is denoted as S_n^m. Let us remark that in the tunneling regime we recover the well-known results for both the background current and Shapiro step heights [13].

In order to illustrate the general results, in Fig. 1 we show the dc current, background current plus Shapiro steps, for different values of α and a frequency $\omega_r = 0.5\Delta$. We can see the two main features that are the subject of the rest of the Letter: (i) the *subharmonic Shapiro steps* S_n^m, with $n \neq 1$, are clearly visible at high transmissions and (ii) the background current exhibits a subharmonic gap structure at voltages $eV = (2\Delta + k\hbar\omega_r)/n$, with n, k integers, which is specially pronounced at low transmissions [14].

Let us start by analyzing the background current. In Figs. 2(c) and 2(d) we show the background current for two different frequencies at a moderate power, $\alpha = 1.0$. The current in the absence of radiation is also shown for comparison. As mentioned above, the most prominent feature in the background current is the appearance of a pronounced subgap structure at voltages $eV = (2\Delta + k\hbar\omega_r)/n$. This structure is specially clear at low transmissions [see Fig. 2(d)] and progressively disappears as the transparency is increased. Indeed, this peculiar subharmonic gap structure was already observed in several experiments in the early 1970s in point contacts and thin-film microbridges [11]. At that time no consistent explanation was given, but it is clear that this structure can be explained in terms of photon-assisted MARs. A step at $eV = (2\Delta + k\hbar\omega_r)/n$ is simply due to the opening of a MAR of order n in which k photons in total are absorbed (k negative) or emitted (k positive). This is illustrated in the upper panels of Fig. 2. In order to understand how this subharmonic structure evolves with the rf power, one can do a systematic perturbative expansion in the transmission. This analysis tells us that at low transparency the height of a current jump at $eV = (2\Delta + k\hbar\omega_r)/n$ is proportional to $J_k^2(n\alpha)$, which is valid as long as $\hbar\omega_r \ll 2\Delta/n$. This result coincides with the phenomenological functional form that was used to fit the experiments by Soerensen *et al.* [11].

Let us now discuss the Shapiro steps. In this case the most important aspect is the existence of subharmonic steps absent in tunnel junctions. These steps arise from the phase locking between the harmonics of the Josephson frequency and the harmonics of the ac radiation. Early experiments on the ac Josephson effect in weak links observed subharmonic steps in the I-V curves [15]. More recently, there have been reported observations of noninteger Shapiro steps in high-T_C contacts [16], S-semiconductor-S junctions [17], and diffusive S-N-S systems [18]. Although the Shapiro steps can be understood as a simple consequence of a nonsinusoidal current-phase relation, the present approach goes beyond a simple "adiabatic" approximation and provides the first microscopic theory of Shapiro steps in contacts of arbitrary transmission. The adiabatic approximation, which introduces the time dependence into the zero bias supercurrent through the Josephson relation, gives rise to the well-known Bessel-function-like behavior of the steps and gives a good description of the tunnel regime [13]. However, as we show

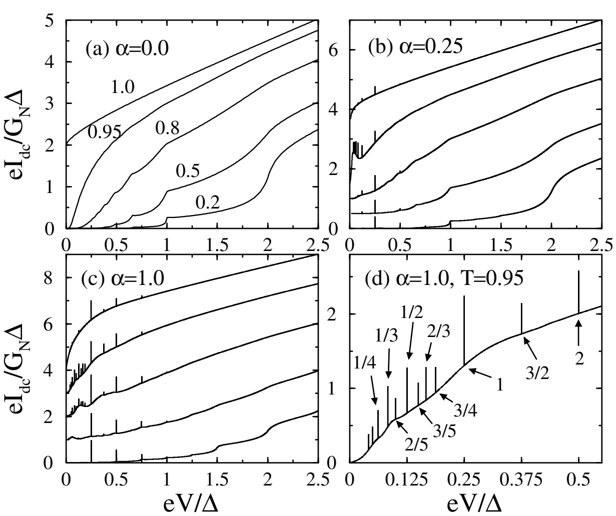

FIG. 1. Zero temperature dc current, I_{dc}, as a function of voltage for a frequency $\omega_r = 0.5\Delta$ and several values of α. The different curves in each panel correspond to different transmissions, as indicated in panel (a). In panels (b) and (c) the curves have been vertically displaced. Panel (d) shows in detail the curve $\mathcal{T} = 0.95$ of panel (c). The current is normalized by the normal conductance $G_N = (2e^2/h)\mathcal{T}$.

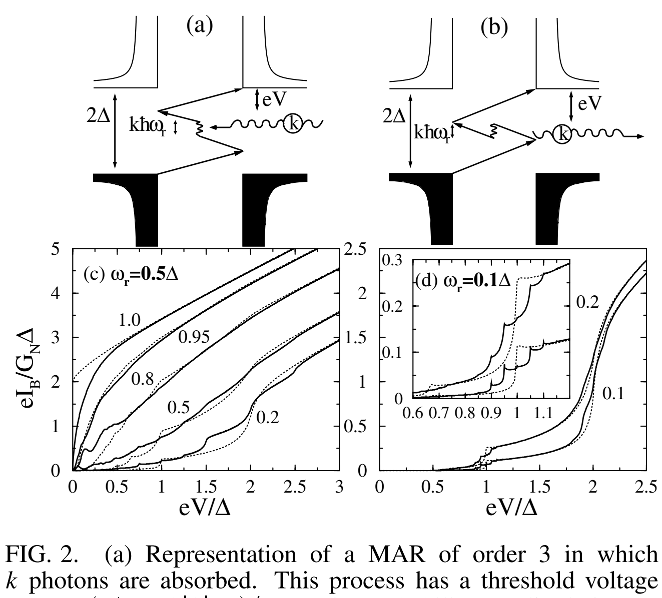

FIG. 2. (a) Representation of a MAR of order 3 in which k photons are absorbed. This process has a threshold voltage $eV_{\rm th} = (2\Delta - \hbar|k|\omega_r)/3$, and its probability amplitude is proportional to $J_k(\alpha)$. (b) A 3-order MAR mediated by the emission of k photons, which contributes to the subgap structure at $eV_{\rm th} = (2\Delta + \hbar|k|\omega_r)/3$. (c) Background current as a function of voltage for $\omega_r = 0.5\Delta$ and different transmissions. (d) The same as in (c) for $\omega_r = 0.1\Delta$. The inset shows a blowup around $eV = \Delta$. The dotted lines in (c) and (d) correspond to the current in the absence of radiation.

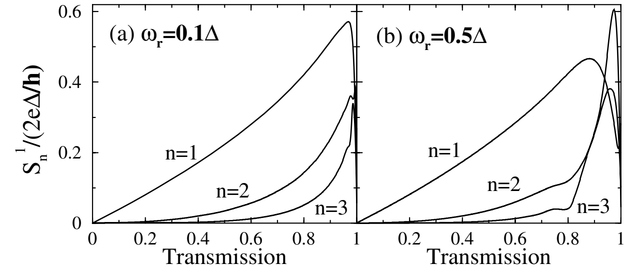

FIG. 3. Shapiro steps S_n^1 versus transmission for $\alpha = 0.25$.

below, such a simple approach fails in the description of a highly transmissive contact.

As a rule of thumb, a Shapiro step S_n^m is visible when the corresponding ac Josephson component, I_n, in the absence of radiation gives a significant contribution. In particular, this means high transmissions (see Figs. 3 and 4 in Ref. [7]). One can show that the leading order in transmission of a Shapiro step S_n^m goes like $\sim\mathcal{T}^n$, which is a consequence of the fact that $I_n \sim \mathcal{T}^n$, and is the reason for the absence of the $n \neq 1$ steps in poorly transmissive contacts. However, near perfect transmission of the subharmonic steps can be even higher than the integer ones. This behavior is illustrated in Fig. 3, where we show the Shapiro steps S_n^1 as a function of the transmission for two different frequencies.

Figure 4 shows the power dependence of the Shapiro steps for a frequency $\omega_r = 0.5\Delta$. Notice that this dependence is rather complicated for both integer and subharmonic steps and clearly deviates from the usual Bessel function behavior. This is due to the frequency dependence of the Josephson components, which is specially pronounced at high transmissions. Neglecting this dependence, i.e., within an adiabatic approximation, one would get that S_n^m evolves as $|J_m(2n\alpha)|$. However, as shown in Fig. 4(b), as the transmission increases the validity of this approximation is restricted to $\alpha \ll 1$. Notice also

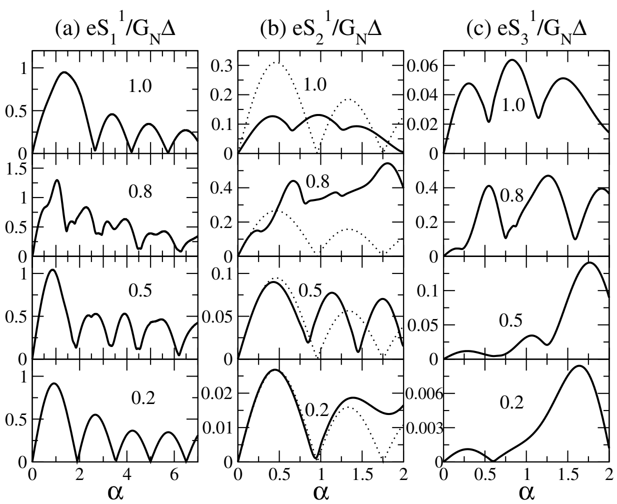

FIG. 4. (a)–(c) Shapiro steps S_n^1 ($n = 1, 2, 3$) as a function of α for $\omega_r = 0.5\Delta$. The dotted lines in panel (b) correspond to the adiabatic approximation: $\sim|J_1(4\alpha)|$.

the complex oscillation pattern at high transmissions (see $\mathcal{T} = 0.8$ curves in Fig. 4), which is due to the fact that several ac components give a significant contribution to the same Shapiro step.

In summary, we have presented a theoretical analysis of the dc current in a superconducting point contact in the presence of a microwave radiation. We have shown that the microscopic theory of coherent multiple Andreev reflections provides a unified description of Shapiro steps and assisted tunneling, explaining in a natural way the observations of subharmonic steps [15–18] and the peculiar subharmonic gap structure under a microwave radiation [11]. Let us finally remark that the results presented in this work are amenable to a quantitative experimental test using atomic-size contacts [2–5].

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