Supplementary materials for "Andreev-Coulomb Drag in Coupled Quantum Dots"

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S-I. THE MODEL

The total Hamiltonian is $\hat{H} = \hat{H}_{\rm dqd} + \hat{H}_{\rm leads} + \hat{H}_t$. The first term is the Hamiltonian for the double quantum dot [Eq. (1) in the main text]:

$$\hat{H}_{\rm dqd} = \sum_{\alpha} \varepsilon_\alpha \hat{n}_\alpha + U_p \hat{n}_{p,\uparrow}\hat{n}_{p,\downarrow} + U_{ap}\hat{n}_a\hat{n}_p. \tag{S1}$$

Here, the number operators are $\hat{n}_p = \Sigma_\sigma \hat{n}_{p,\sigma} = \Sigma_\sigma \hat{d}^\dagger_{p,\sigma}\hat{d}_{p,\sigma}$ with spin $\sigma = \{\uparrow,\downarrow\}$ and $\hat{n}_a = \hat{d}^\dagger_a \hat{d}_a$ where $\hat{d}_{p,\sigma}$ and $\hat{d}_a$ denote the electron annihilation operators in passive ($\alpha = p$) and active ($\alpha = a$) dots with energy ε_α (we take spinless electrons for $\alpha = a$ since the spin degree of freedom in the active subsystem does not change the main physics of the problem). The Hamiltonian of the electrodes is given by

$$\hat{H}_{\rm leads} = \sum_k \varepsilon_{k,L}\hat{c}^\dagger_{k,L}\hat{c}_{k,L} + \sum_k \varepsilon_{k,R}\hat{c}^\dagger_{k,R}\hat{c}_{k,R} + \sum_{k,\sigma}\varepsilon_{k,N}\hat{c}^\dagger_{k,N,\sigma}\hat{c}_{k,N,\sigma} + \sum_{k,\sigma}\varepsilon_{k,S}\hat{c}^\dagger_{k,S,\sigma}\hat{c}_{k,S,\sigma} + \sum_k \Delta\left(\hat{c}^\dagger_{k,S,\uparrow}\hat{c}^\dagger_{k,S,\downarrow} + \text{h.c.}\right), \tag{S2}$$

where $\hat{c}_k$ is the annihilation operator for electrons with energy ε_k and Δ is the order parameter in the superconducting lead. The passive dot is coupled to normal (N) and superconductor (S) electrodes. In the drag configuration, these two electrodes have the same chemical potential μ. Without loss of generality, we take the energy reference at $\mu_N = \mu_S = 0$. Furthermore, the active dot is attached to two normal electrodes (L and R) through which a symmetric bias voltage is applied as $\mu_L = -\mu_R = eV_{\rm bias}/2$, where e is the electron charge. As a consequence, the tunneling Hamiltonian becomes

$$\hat{H}_t = \sum_k \left(t_L \hat{d}^\dagger_a \hat{c}_{k,L} + t_R \hat{d}^\dagger_a \hat{c}_{k,R} + \text{h.c.}\right) + \sum_{k,\sigma}\left(t_N \hat{d}^\dagger_{p,\sigma}\hat{c}_{k,N,\sigma} + t_S \hat{d}^\dagger_{p,\sigma}\hat{c}_{k,S,\sigma} + \text{h.c.}\right), \tag{S3}$$

where t are the dot-lead tunnel couplings. In the following, we consider the wide-band approximation, in which case the tunnel hybridization strength is given by $\Gamma_\beta = 2\pi|t_\beta|^2\rho_0^\beta$ for $\beta = L, R, N, S$, where ρ_0^β is the corresponding electrode's density of states in its normal state.

S-II. NONEQUILIBRIUM GREEN'S FUNCTIONS METHOD

Here, we give the details of calculating the drag current using the nonequilibrium Green's functions (NEGF) formalism [S1]. We consider the Hamiltonian of noninteracting double quantum dots, $\hat{H}_{\rm dqd}(U_p = U_{ap} = 0)$, and the Hamiltonian of electrodes, $\hat{H}_{\rm leads}$, as the unperturbed Hamiltonian and proceed by considering $\hat{H}_t$ and $\hat{H}_{\rm int} = U_{ap}\hat{n}_a\hat{n}_p + U_p\hat{n}_{p,\uparrow}\hat{n}_{p,\downarrow}$ the interaction Hamiltonians. Then, we define the contour-ordered single particle Green's function of the system for the active and passive dots, respectively, by

$$iG^c_a(\tau,\tau') = \left\langle T_c \hat{d}_a(\tau)\,\hat{d}^\dagger_a(\tau')\right\rangle, \tag{S4}$$

$$i\boldsymbol{G}^c_p(\tau,\tau') = \left\langle T_c \hat{\Psi}_p(\tau)\,\hat{\Psi}^\dagger_p(\tau')\right\rangle, \tag{S5}$$

where $\langle\ldots\rangle$ is the ground state expectation value of the interacting system, T_c is the time-ordering operator along the Keldysh contour and τ and τ' are time variables along the Keldysh contour. In Eq. (S5), we represente the Green's function of the passive dot in the Nambu basis defined by $\hat{\Psi}^\dagger_p = (\hat{d}^\dagger_{p,\uparrow}, \hat{d}_{p,\downarrow})$. Here and in the following, we show the quantities in the Nambu basis using bold letters such as $\boldsymbol{G}$ and $\boldsymbol{\Sigma}$.

A. Dyson equation for the passive dot

In frequency space, the retarded interacting Green's function of the passive dot can be obtained from

$$\boldsymbol{G}_p^R(\omega) = \left\{ \left[\boldsymbol{g}_p^R(\omega)\right]^{-1} - \boldsymbol{\Sigma}_{p,\text{int}}^R(\omega) \right\}^{-1}, \tag{S6}$$

where $\boldsymbol{g}_p^R(\omega)$ is the mean-field retarded Green's function given by

$$\boldsymbol{g}_p^R(\omega) = \left(\omega\mathbf{I} - \boldsymbol{h}_p - \boldsymbol{\Sigma}_{p,\text{leads}}^R\right)^{-1}, \tag{S7}$$

where $\mathbf{I}$ is the 2×2 identity matrix and $\boldsymbol{h}_p$ is a diagonal matrix with diagonal elements $(\varepsilon_p+U_{ap}\langle\hat{n}_a\rangle+U_p\langle\hat{n}_{p,\downarrow}\rangle, -\varepsilon_p-U_{ap}\langle\hat{n}_a\rangle-U_p\langle\hat{n}_{p,\uparrow}\rangle)$. Moreover, $\boldsymbol{\Sigma}_{p,\text{leads}}^R(\omega) = \boldsymbol{\Sigma}_{p,N}^R(\omega)+\boldsymbol{\Sigma}_{p,S}^R(\omega)$ is the sum of self-energies due to coupling the passive dot to the normal and superconducting electrodes. Their respective expressions in the wide-band approximation read

$$\boldsymbol{\Sigma}_{p,N}^R(\omega) = -i\Gamma_{p,N}\mathbf{I}, \tag{S8}$$

and

$$\boldsymbol{\Sigma}_S^R(\omega) = -i\Gamma_S\beta(\omega)\begin{pmatrix} 1 & -\frac{\Delta}{\omega} \\ -\frac{\Delta}{\omega} & 1 \end{pmatrix}, \tag{S9}$$

where $\beta(\omega)$ is given by

$$\beta(\omega) = \frac{|\omega|}{\sqrt{\omega^2-\Delta^2}}\theta(|\omega|-\Delta) - i\frac{\omega}{\sqrt{\Delta^2-\omega^2}}\theta(\Delta-|\omega|), \tag{S10}$$

with $\theta(\ldots)$ the Heaviside step function. Furthermore, in Eq. (S6), $\boldsymbol{\Sigma}_{p,\text{int}}^R(\omega)$ is the retarded self-energy of the passive dot due to interaction with the active dot. Its expression will be given later.

We also need to calculate the interacting lesser and greater Green's functions of the passive dot, which are given by

$$\boldsymbol{G}_p^{<,>}(\omega) = \boldsymbol{G}_p^R(\omega)\left(\boldsymbol{\Sigma}_{p,\text{leads}}^{<,>}(\omega) + \boldsymbol{\Sigma}_{p,\text{int}}^{<,>}(\omega)\right)\boldsymbol{G}_p^A(\omega), \tag{S11}$$

where the advanced Green's function is obtained as $\boldsymbol{G}_p^A(\omega) = \left[\boldsymbol{G}_p^R(\omega)\right]^\dagger$ and $\boldsymbol{\Sigma}_{p,\text{leads}}^{<,>}$ can be calculated from

$$\boldsymbol{\Sigma}_{p,N/S}^< = -2\,\text{Im}\left(\boldsymbol{\Sigma}_{p,N/S}^R\right) f_{N/S}(\omega), \tag{S12}$$

and

$$\boldsymbol{\Sigma}_{p,N/S}^> = -2\,\text{Im}\left(\boldsymbol{\Sigma}_{p,N/S}^R\right)[1-f_{N/S}(\omega)]. \tag{S13}$$

$f_\beta(\omega) = \{1+\exp[(\omega-\mu_\beta)/k_BT]\}^{-1}$ is the Fermi distribution function of electrode $\beta = N, S$ with chemical potential μ_β and temperature T. The expressions for the lesser and greater interacting self-energies, $\boldsymbol{\Sigma}_{p,\text{int}}^{<,>}$, are given below.

B. Dyson equation for the active dot

Next, we will focus on the active dot. Its full retarded Green's function is given by

$$G_a^R(\omega) = \left\{ \left[g_a^R(\omega)\right]^{-1} - \Sigma_{a,\text{int}}^R(\omega) \right\}^{-1}, \tag{S14}$$

where the mean-field retarded Green's function $g_a^R(\omega)$ is

$$g_a^R(\omega) = \left(\omega - \varepsilon_a - U_{ap}\sum_\sigma\langle\hat{n}_{p,\sigma}\rangle - \Sigma_{a,\text{leads}}^R\right)^{-1}. \tag{S15}$$

Here, $\Sigma_{a,\text{leads}}^R(\omega) = \Sigma_{a,L}^R(\omega) + \Sigma_{a,R}^R(\omega)$ is the self-energy due to coupling the active dot to its right and left normal metal electrodes: $\Sigma_{a,\beta}^R(\omega) = -i\Gamma_{a,\beta}$ for $\beta = R, L$. The lesser Green's function of the active dot is also given in a similar manner to the passive dot in Eq. (S11) by replacing subscript p with a and considering the hybridization of the active dot with two normal metallic electrodes.

C. Interaction self-energies for the passive dot

We obtain the interacting lesser and greater self-energies for the passive dot within second-order perturbation theory [S2]. We generalize the results of Ref. [S3] to include the nonlocal capacitive interaction between the passive and active dots:

$$\begin{aligned}\boldsymbol{\Sigma}_{p,\mathrm{int}}^{<,>}(\omega) &= \left(\frac{U_p}{2\pi}\right)^2 \int d\omega_1 Q_p^{<,>}(\omega)\,\hat{\sigma}_y \left[\boldsymbol{G}_p^{>,<}(\omega_1-\omega)\right]^T \hat{\sigma}_y \\ &+ \left(\frac{U_{ap}}{2\pi}\right)^2 \int d\omega_1 \hat{\sigma}_z \boldsymbol{G}_p^{<,>}(\omega_1)\,\hat{\sigma}_z W_a^{<,>}(\omega-\omega_1)\,,\end{aligned} \tag{S16}$$

where $\hat{\sigma}_y$ and $\hat{\sigma}_z$ are the second and third Pauli matrices, Q_p reads

$$Q_p^{<,>}(\omega) = \int d\omega_1 \left[G_{p,11}^{<,>}(\omega_1)\,G_{p,22}^{<,>}(\omega-\omega_1) - G_{p,12}^{<,>}(\omega_1)\,G_{p,21}^{<,>}(\omega-\omega_1)\right], \tag{S17}$$

and W_a is given by

$$W_a^{<,>}(\omega) = \int d\omega_1 G_a^{<,>}(\omega_1)\,G_a^{>,<}(\omega_1-\omega)\,. \tag{S18}$$

The interacting retarded self-energy of the passive dot is

$$\begin{aligned}\boldsymbol{\Sigma}_{p,\mathrm{int}}^{R}(\omega) &= \left(\frac{U_p}{2\pi}\right)^2 \int d\omega_1 \left[Q_p^{<}(\omega)\,\hat{\sigma}_y \left[\boldsymbol{G}_p^{A}(\omega_1-\omega)\right]^T \hat{\sigma}_y + Q_p^{R}(\omega)\,\hat{\sigma}_y \left[\boldsymbol{G}_p^{<}(\omega_1-\omega)\right]^T \hat{\sigma}_y\right] \\ &+ \left(\frac{U_{ap}}{2\pi}\right)^2 \int d\omega_1 \left[\hat{\sigma}_z \boldsymbol{G}_p^{<}(\omega_1)\,\hat{\sigma}_z W_a^{R}(\omega-\omega_1) + \hat{\sigma}_z \boldsymbol{G}_p^{R}(\omega_1)\,\hat{\sigma}_z \left[W_a^{<}(\omega-\omega_1) + W_a^{R}(\omega-\omega_1)\right]\right],\end{aligned} \tag{S19}$$

where

$$\begin{aligned}Q_p^{R}(\omega) = \int d\omega_1 \big[&G_{p,11}^{<}(\omega_1)\,G_{p,22}^{R}(\omega-\omega_1) - G_{p,12}^{<}(\omega_1)\,G_{p,21}^{R}(\omega-\omega_1) \\ &+ G_{p,11}^{R}(\omega_1)\,G_{p,22}^{<}(\omega-\omega_1) - G_{p,12}^{R}(\omega_1)\,G_{p,21}^{<}(\omega-\omega_1) \\ &+ G_{p,11}^{R}(\omega_1)\,G_{p,22}^{R}(\omega-\omega_1) - G_{p,12}^{R}(\omega_1)\,G_{p,21}^{R}(\omega-\omega_1)\big]\,,\end{aligned} \tag{S20}$$

and

$$W_a^{R}(\omega) = \int d\omega_1 \left[G_a^{<}(\omega_1)\,G_a^{A}(\omega_1-\omega) + G_a^{R}(\omega_1)\,G_a^{<}(\omega_1-\omega)\right]. \tag{S21}$$

D. Interacting self-energies for the active dot

The interacting lesser and greater self-energies for the active dot are given by

$$\Sigma_{a,\mathrm{int}}^{<,>}(\omega) = \left(\frac{U_{ap}}{2\pi}\right)^2 \int d\omega_1 G_a^{<,>}(\omega_1)\,W_p^{<,>}(\omega-\omega_1)\,, \tag{S22}$$

where the W_p functions are

$$W_p^{<,>}(\omega) = \int d\omega_1 \mathrm{Tr}\left[\hat{\sigma}_z \boldsymbol{G}_p^{<,>}(\omega_1)\,\hat{\sigma}_z \boldsymbol{G}_p^{>,<}(\omega_1-\omega)\right], \tag{S23}$$

$$W_p^{R}(\omega) = \int d\omega_1 \mathrm{Tr}\left[\hat{\sigma}_z \boldsymbol{G}_p^{<}(\omega_1)\,\hat{\sigma}_z \boldsymbol{G}_p^{A}(\omega_1-\omega) + \hat{\sigma}_z \boldsymbol{G}_p^{R}(\omega_1)\,\hat{\sigma}_z \boldsymbol{G}_p^{<}(\omega_1-\omega)\right], \tag{S24}$$

$\mathrm{Tr}[\ldots]$ being the trace over the Nambu matrices. The interacting retarded self-energy of the active dot is given by

$$\Sigma_{a,\mathrm{int}}^{R}(\omega) = \left(\frac{U_{ap}}{2\pi}\right)^2 \int d\omega_1 \left[G_a^{<}(\omega_1)\,W_p^{R}(\omega-\omega_1) + G_a^{R}(\omega_1)\left[W_p^{<}(\omega-\omega_1) + W_p^{R}(\omega-\omega_1)\right]\right]. \tag{S25}$$

E. Expression for the current

The above discussion provides a complete description of the required equations to calculate the interacting Green's functions of both active and passive dots in the nonequilibrium steady state. In our numerical calculations, we have performed self-consistent calculations to obtain the self-consistent Green's functions and self-energies of the system. Once the NEGFs are obtained the electric current through the active dot to the lead $\beta = L, R$ can be calculated as [S1]

$$I_{a,\beta} = \frac{e}{\hbar}\int \frac{d\omega}{2\pi}\left[G_a^{<}(\omega)\,\Sigma_{a,\beta}^{>}(\omega) - G_a^{>}(\omega)\,\Sigma_{a,\beta}^{<}(\omega)\right]. \tag{S26}$$

On the other hand, the electric current through the passive dot to the lead $\beta = N, S$ can be evaluated from $I_{p,\beta} = \int d\omega \mathcal{I}_\beta(\omega)$, where $\mathcal{I}_\beta(\omega)$ is the energy resolved current density [S4]

$$\mathcal{I}_\beta(\omega) = \frac{e}{2h}\mathrm{Tr}\left[\hat{\sigma}_z(\boldsymbol{G}_p^R(\omega)\,\boldsymbol{\Sigma}_{p,\beta}^{<}(\omega) + \boldsymbol{G}_p^{<}(\omega)\,\boldsymbol{\Sigma}_{p,\beta}^{A}(\omega) - \boldsymbol{\Sigma}_{p,\beta}^{R}(\omega)\,\boldsymbol{G}_p^{<}(\omega) - \boldsymbol{\Sigma}_{p,\beta}^{<}(\omega)\,\boldsymbol{G}_p^{A}(\omega)\right]. \tag{S27}$$

This expression for the electric current of the passive dot has the advantage that by taking appropriate integration limits we can quantitatively distinguish between the current in the subgap domain (I_A), which is mainly due to pair tunneling accompanied by Andreev reflection, and the current outside the superconductor gap (I_{qp}), which is dominated by quasiparticle tunneling into the superconductor continuum. It is thus clear that the total current flowing through the passive dot can be rewritten as $I_p = I_A + I_{qp}$ where

$$I_A = \int_{-\Delta}^{\Delta}\mathcal{I}(\omega)\,d\omega, \tag{S28}$$

and

$$I_{qp} = \left(\int_{-\infty}^{-\Delta} + \int_{\Delta}^{\infty}\right)\mathcal{I}(\omega)\,d\omega. \tag{S29}$$

S-III. RATE EQUATION METHOD

The rate equation method can be employed in the $k_{\mathrm{B}}T \gg \Gamma_\beta$ regime to calculate the electric current using the eigenstates and eigenenergies of $\hat{H}_{\mathrm{dqd}}$ as follows [S5]

$$I_\alpha = \frac{e}{\hbar}\sum_{\lambda,\kappa}\left(\gamma_{\lambda\kappa}^{\alpha N} - \Gamma_{\lambda\kappa}^{\alpha N}\right)P(\kappa), \tag{S30}$$

where $\alpha = p, a$ as before and

$$\Gamma_{\lambda\kappa}^{\alpha\beta} = \Gamma_\beta|\langle\lambda|\hat{\delta}_\alpha^\dagger|\kappa\rangle|^2 f_\beta(E_\lambda - E_\kappa), \tag{S31}$$

$$\gamma_{\lambda\kappa}^{\alpha\beta} = \Gamma_\beta\left|\left\langle\lambda\middle|\hat{\delta}_\alpha\middle|\kappa\right\rangle\right|^2[1 - f_\beta(E_\kappa - E_\lambda)], \tag{S32}$$

are the tunneling rates in and out of the dot α, from and into electrode $\beta = N, L, R, S$, respectively. In the above equations, $\hat{\delta}_a \equiv \hat{d}_a$ and $\hat{\delta}_p \equiv \sum_\sigma \hat{d}_{p,\sigma}$. Moreover, $|\lambda\rangle$ is the eigenstate of $\hat{H}_{\mathrm{dqd}}$ with energy E_λ. In Eq. (S30), $P(\lambda)$ is the occupation probability for state $|\lambda\rangle$ which is calculated by solving the system of equations $0 = \sum_{\alpha,\beta,\kappa}\left(\Gamma_{\kappa\lambda}^{\alpha\beta}P(\lambda) - \gamma_{\lambda\kappa}^{\alpha\beta}P(\kappa)\right)$ together with the normalization condition $\sum_\kappa P(\kappa) = 1$.

A. Current carried by Cooper pairs

To investigate the drag current corresponding to the subgap electron transport in the passive dot, it is useful to take the infinite-gap approximation. This approach is equivalent to taking the superconductor energy gap to be the largest energy scale in the system. Hence, the coupling between the superconducting electrode and the passive dot can be replaced with an effective pairing term $\Gamma_S(\hat{d}^\dagger_{p,\uparrow}\hat{d}^\dagger_{p,\downarrow}+\text{h.c.})$ in the Hamiltonian of the passive dot [S6, S7]. As a consequence, the Hamiltonian of the isolated double quantum dot can be exactly diagonalized as

$$\begin{aligned} |\sigma,n\rangle\,, \qquad\qquad & E_{\sigma,n}=\varepsilon_p+n(\varepsilon_a+U_{ap}),\\ |\pm,n\rangle=\mathcal{N}^{-1}_{\pm,n}(A_{\pm,n}\,|0,n\rangle-\Gamma_S\,|2,n\rangle), \qquad & E_{\pm,n}=n\varepsilon_a+A_{\mp,n}, \end{aligned} \tag{S33}$$

where the first argument in the kets (0, σ or 2) represents the state of the passive dot while the second ($n=0,1$) is the occupation of the active dot. Moreover, $\mathcal{N}_{\pm,n}$ is a normalization factor and

$$A_{\pm,n}=\tilde{\varepsilon}_n\pm\sqrt{(\tilde{\varepsilon}_n)^2+\Gamma_S^2}, \tag{S34}$$

where $\tilde{\varepsilon}_n=\varepsilon_p+U_p/2+nU_{ap}$. The above eigensystem is composed of eight states which allows us to employ Eq. (S30) to calculate the electric current through the passive and active dots in the infinite gap approximation.

1. *Tunneling rates and electron-hole symmetry*

The number of electrons in the passive dot is not well defined when it is in one of the superposition states $|\pm,n\rangle$. Then, transitions to odd states $|\sigma,n\rangle$ may involve an electron either tunneling in or out of the dot. Their probabilities are determined by the tunneling rates $\Gamma^{\alpha\beta}_{\lambda\kappa}=\mathcal{G}^{\alpha\beta}_{\lambda\kappa}f_\beta(E_\lambda-E_\kappa)$ for transitions involving an electron tunneling from terminal β into quantum dot α, and $\gamma^{\alpha\beta}_{\lambda\kappa}=\mathcal{J}^{\alpha\beta}_{\lambda\kappa}[1-f_\beta(E_\kappa-E_\lambda)]$ for those involving an electron tunneling out of the dot. Here, $\mathcal{G}^{\alpha\beta}_{\lambda\kappa}=\Gamma_\beta|\langle\lambda|\hat{\delta}^\dagger_\alpha|\kappa\rangle|^2$ and $\mathcal{J}^{\alpha\beta}_{\lambda\kappa}=\Gamma_\beta|\langle\lambda|\hat{\delta}_\alpha|\kappa\rangle|^2$.

For the transition $|\sigma,n\rangle\to|+,n\rangle$, we have

$$\mathcal{G}^{p,N}_{+n,\sigma n}=\Gamma_N\mathcal{N}^{-2}_{+n}\Gamma_S^2 \tag{S35}$$

$$\mathcal{J}^{p,N}_{+n,\sigma n}=\Gamma_N\mathcal{N}^{-2}_{+n}\left(\tilde{\varepsilon}_n+\sqrt{\tilde{\varepsilon}_n^2+\Gamma_S^2}\right)^2. \tag{S36}$$

Whether this transition is more likely to happen with an electron tunneling in or out of the passive dot depends on the sign of $\tilde{\varepsilon}_n$. The two rates are equal at $\tilde{\varepsilon}_n=0$.

For $|-,n\rangle\to|\sigma,n\rangle$, we have

$$\mathcal{G}^{p,N}_{\sigma n,-n}=\Gamma_N\mathcal{N}^{-2}_{-n}\left(-\tilde{\varepsilon}_n+\sqrt{\tilde{\varepsilon}_n^2+\Gamma_S^2}\right)^2 \tag{S37}$$

$$\mathcal{J}^{p,N}_{\sigma n,-n}=\Gamma_N\mathcal{N}^{-2}_{-n}\Gamma_S^2. \tag{S38}$$

Note that at the point $\tilde{\varepsilon}_0+\tilde{\varepsilon}_1=0$, the electron-hole symmetry is established by having $\mathcal{G}^{p,N}_{\pm1,\sigma1}=\mathcal{J}^{p,N}_{\sigma0,\mp0}$. Hence, at this point, sequences of the form $|\sigma,0\rangle\to|+,0\rangle\to|-,1\rangle\to|\sigma,1\rangle$ contribute on average to the transport of an electron and of a hole. The same (though with opposite contributions) is valid for the cycle $|\sigma,1\rangle\to|+,1\rangle\to|-,0\rangle\to|\sigma,0\rangle$, resulting in no drag current.

2. *NEGF vs. rate equation results*

Our NEGF formalism is capable of considering the coupling between the passive dot and the leads nonperturbatively. However, it is expected that in the weak tunneling regime, where $k_BT\gg\Gamma_\beta$, the NEGF results reproduce the results of the rate equation method.

The infinite-gap approximation which we discussed earlier in this section can be explored within the NEGF formalism by setting a large value to Δ in Eq. (S9). In Fig. S1, we compare the results obtained from both methods. In Fig. S1(a), we consider a large bias voltage on the active dot and plot the drag current as a function of ε_p. We can see that the NEGF results are in good agreement with those of the rate equation method, especially the sign of the drag current as a function of ε_p, and the inversion of the drag current at the point $\varepsilon_p=-(U_{ap}+U_p)/2\equiv-U_{\text{total}}/2$. In Fig. S1(b), we take $\varepsilon_p=-0.7U_{\text{total}}$, and plot the drag current as a function of bias voltage on the active dot, where we find good agreement between the results obtained from both methods. The insets in both panels show the corresponding results for the case where $k_BT\sim\Gamma_N$. We can see that by decreasing the temperature the rate equation results depart from the NEGF ones, which is an indication that in this parameter regime the results from rate equation method are not quantitatively correct, though they still give a proper qualitative behaviour.

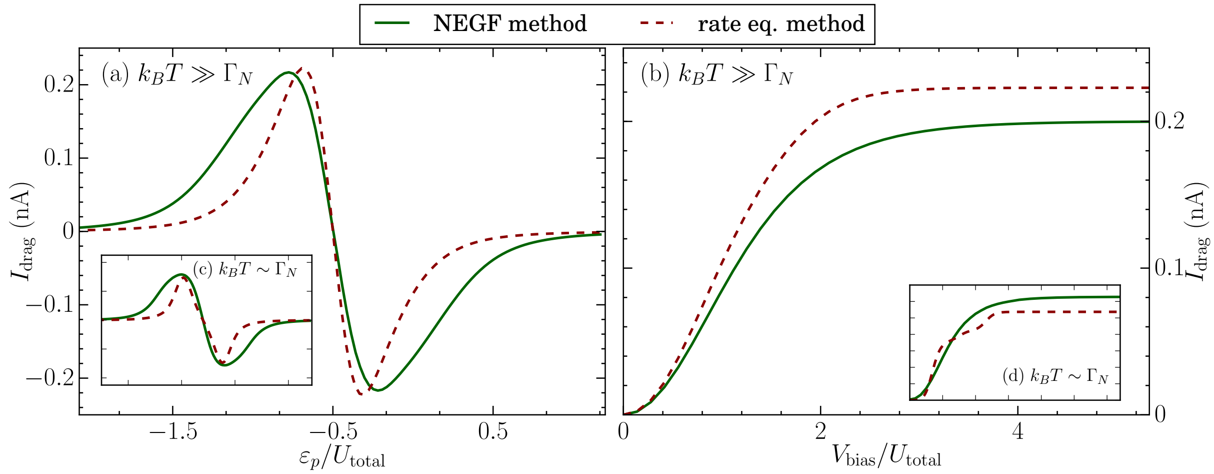

FIG. S1. Comparison of NEGF results with the results obtained from rate equation method in the infinite gap approximation. Left panels show drag current as a function of ε_p for a large $V_{\rm bias}$ while right panels show drag current as a function of $V_{\rm bias}$ for $\varepsilon_p = -0.7U_{\rm total}$. In (a) and (b) we take $k_BT = 1$ meV whereas in (c) and (d) $k_BT = 0.03$ meV. Here, Δ in the NEGF calculations is set to a large value ($\Delta = 10$ meV). Additional parameters: $U_p = 5U_{ap} = 0.5$ meV and $\Gamma_L = \Gamma_R = \Gamma_N = \Gamma_S/3 = 0.05$ meV.

B. Current carried by quasiparticles

Let us now consider the case when the contribution of Cooper pairs is negligible. The only effect of the superconductor is then introduced by the presence of a gap in the density of states. Tunneling through the S barrier is only possible for electrons with energy falling outside the gap region.

Assuming strong on-site Coulomb interactions, there are only four relevant charge states, described in terms of the charge occupations: $|n_p, n_a\rangle$, with $n_\alpha = 0, 1$. We ignore the spin degree of freedom here, for simplicity. The different tunneling rates in the passive dot are:

$$\Gamma^{p,N}_{1n,0n} = \Gamma_N f_{Nn}, \tag{S39}$$

for electrons tunneling in from terminal N with the active dot having n electrons, and

$$\Gamma^{p,S}_{1n,0n} = \Gamma^{\rm qp}_{Sn} f_{Sn} = \Gamma_S \nu_n f_{Sn}, \tag{S40}$$

for electrons tunneling in from terminal S. For transitions into the active dot, we have $\Gamma^{a,\beta}_{n1,n0} = \Gamma_\beta f_{\beta n}$. Here, $f_{\beta n} = f_\beta(\varepsilon_p + nU_{ap})$, when $\beta = N, S$, and $f_{\beta n} = f_\beta(\varepsilon_a + nU_{ap})$, when $\beta = L, R$. For electrons tunneling out to the respective terminals, we need to make the replacement $f_{\beta n} \to 1 - f_{\beta n}$. Since the passive system is unbiased, we define $f_n \equiv f_{Nn} = f_{Sn}$. This way, tunneling events involving the superconductor depend on the occupation of the active dot. Note that we are assuming the simplest case where Γ_N and Γ_S are energy-independent, which emphasizes the key role of the gap.

Writing down a rate equation for these states [S8], we find the drag current

$$I_{\rm drag} = -e(\nu_0 - \nu_1)\frac{\Gamma_N\Gamma_S}{\hbar\gamma^3}\sum_{\beta,\beta'=L,R}\mathcal{A}_{\beta\beta'}\Gamma_\beta\Gamma_{\beta'}, \tag{S41}$$

with the prefactor $\gamma^3 > 0$ setting the normalization of the steady-state density matrix, and

$$\mathcal{A}_{\beta\beta'} = f_0 f_{\beta 1}(1 - f_{\beta 0}) - f_1 f_{\beta 0}(1 - f_{\beta' 1}) + \frac{1}{2}(f_{\beta 0} + f_{\beta' 0} - f_{\beta 1} - f_{\beta' 1}). \tag{S42}$$

Remarkably, the presence of a drag current only relies on the asymmetry introduced by the gap, $\nu_0 - \nu_1$.

We obtain a simpler expression in the high bias limit, where $f_{Ln} \to 1$ and $f_{Rn} \to 0$, hence tunneling is unidirectional in the active system and:

$$I_{\rm drag} = -e\frac{(\nu_0 - \nu_1)(f_0 - f_1)\Gamma_N\Gamma_S\Gamma_L\Gamma_R}{(\Gamma_L + \Gamma_R)\left\{\Gamma_N[\Gamma_N + (\nu_0 + \nu_1)\Gamma_S + \Gamma_L + \Gamma_R] + \Gamma_S(\nu_0\nu_1\Gamma_S + \nu_1\Gamma_L + \nu_0\Gamma_R)\right\}}. \tag{S43}$$

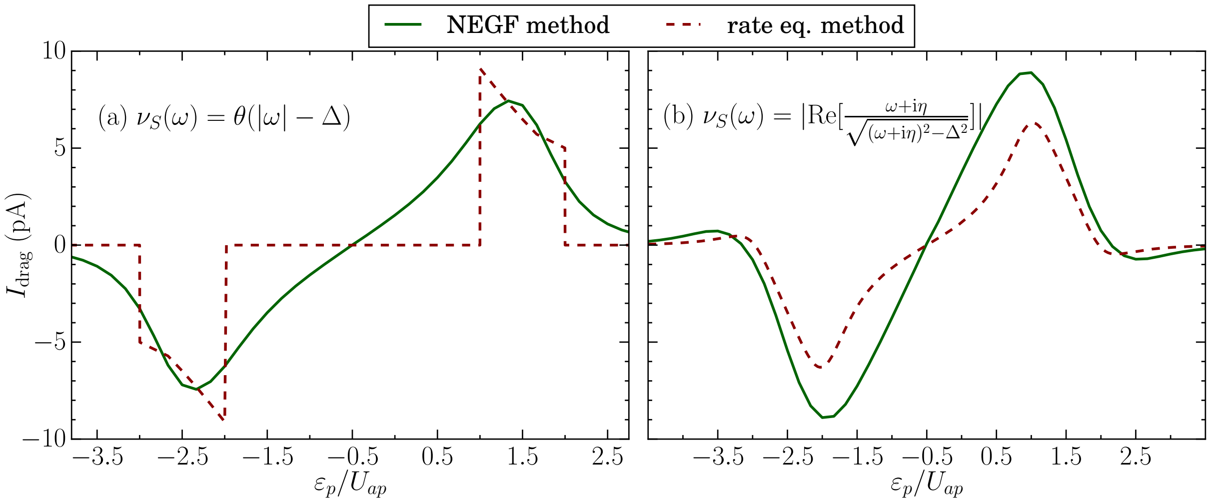

FIG. S2. Drag current as a function of ε_p calculated using the NEGF (solid line) and the rate equation (dashed line) methods, when the superconductor lead is replaced by a normal metal with a modified density of states, $\nu_S(\omega)$. We take in (a) a flat pseudo-gap and in (b) a Dynes density of states with $\eta = \Delta/6$ for the normal lead. Additional parameters: $\Delta = 1$ meV, $\Gamma_L = \Gamma_R = \Gamma_N = \Gamma_S = 0.0016$ meV, $U_p = 0$ and $U_{ap} = k_B T = 0.5$ meV.

We find a simple interpretation from the above expression for the drag current by introducing a hard gap in the density of states $\nu_n = g(\varepsilon_p + nU_{ap})\theta\left(|\varepsilon_p + nU_{ap}| - \Delta\right)$. For the moment, we ignore the explicit energy dependence of the function $g(E)$. Now, consider for example that the passive dot level lies in the gap of the superconductor, $-\Delta < \varepsilon_p < \Delta$, but $\varepsilon_p + U_{ap} > \Delta$. Then, $\nu_0 = 0$ and $\nu_1 = 1$. Further, consider for simplicity that $k_{\mathrm{B}}T \ll U_{ap}$, thereby we can approximate $f_0 \to 1$ and $f_1 \to 0$. Then, the only possible way to charge the passive dot is by an electron tunneling from N when the active dot is empty. Due to the gap, this electron cannot tunnel out until the active dot becomes charged, in which case it can tunnel over the gap into the superconductor. This sequence is completed when the active dot returns to its empty state, hence generating the drag effect. Note that the electron in the passive dot might also tunnel back to N, in this case not contributing to the current. For finite temperatures, this sequence is still the dominant process as $f_0 - f_1 > 0$. Therefore, the drag current is positive.

The level position with respect to the gap changes the sign of the current. In the opposite case when $-\Delta < \varepsilon_p + U_{ap} < \Delta$ and $\varepsilon_p < -\Delta$ (i.e., $\nu_0 \neq 0$ and $\nu_1 = 0$), the passive dot can be charged both from N and S, but it can only tunnel out to N, hence leading to a negative drag current. If both energies lie in the gap, then $\nu_0 = \nu_1 = 0$ and there is no drag current.

Thus, for levels close to the Fermi energy the direction of the current is strongly dominated by the gap. Only in the case where both energies fall outside the gap will the sign of the current depend on details of the density of states through the function $g(E)$.

1. *NEGF vs. rate equation results*

We can also compare the quasiparticle drag current obtained from the rate equation formalism with the NEGF results. To this end, we replace the superconductor lead self-energy in Eq. (S9) with a normal lead self-energy as in Eq. (S8) multiplied by an energy dependent density of states, $\nu_S(\omega)$.

In Fig. S2(a), we show the results obtained from both NEGF and rate equation methods for a hard gap density of states $\nu_S(\omega) = \Theta\left(|\omega| - \Delta\right)$. In this case, the rate equation expression calculated from Eq. (S43) is able to predict the correct sign of the drag current as a function of ε_p, as compared to the NEGF result. However, the rate equation results show features with sharp edges, which differ from the smooth NEGF curves. We can understand this because the rate equation method neglects quantum dot level broadenings due to tunneling. Hence, for piecewise constant pseudo-gap conditions, the drag current is conditioned on ε_p and $\varepsilon_p + U_{ap}$ laying on different parts of the $\nu(E)$ profile (one in and one out of the gap).

In order to correct this and introduce finite quasiparticle lifetimes, one typically considers the Dynes density of states [S9, S10] which is given by

$$\nu(\omega) = \left| \mathrm{Re}\left[\frac{\omega + \mathrm{i}\eta}{\sqrt{(\omega + \mathrm{i}\eta)^2 - \Delta^2}} \right] \right|, \tag{S44}$$

where η is a positive constant. Figure S2(b) shows the drag current obtained from both methods using Eq. (S44). We observe that the sign of the drag current is again correctly reproduced by both methods while the broadening of the drag current is much better in this case. We remark the additional drag current sign change outside the gap due to the energy dependence of the Dynes density of states near the gap edges.

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